Random-Edge is slower than Random-Facet on abstract cubes

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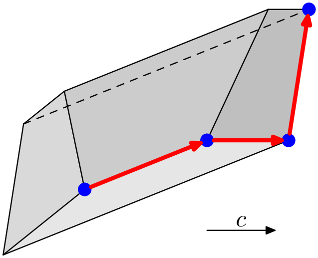
August 13, 2016

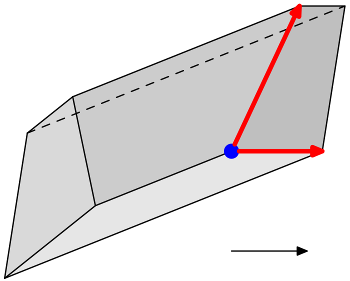
The simplex algorithm, Dantzig [1947]

$$\begin{array}{lrcl} \max & c^T x & & \\ s.t. & Ax & \leq & b \end{array}$$



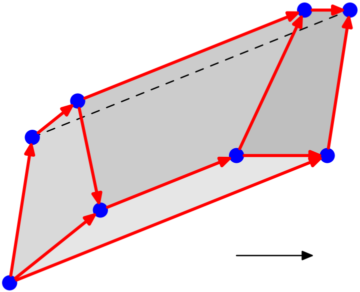
- **Linear programming**: Maximize a **linear objective function** subject to **linear constraints**.
- **The simplex algorithm**: Move from **vertex** to **vertex** along **edges** while improving the objective.
  - This operation is called a **pivot**.

Pivoting rules



- A **pivoting rule** chooses which **improving pivot** to make.
- RANDOM-EDGE: Repeatedly use a uniformly random **improving pivot**.

Orientations



- The objective function defines an **orientation** of the **edges**.
- Many pivoting rules only rely on this orientation.

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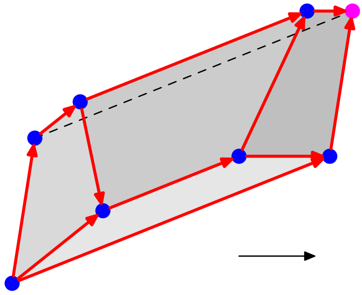
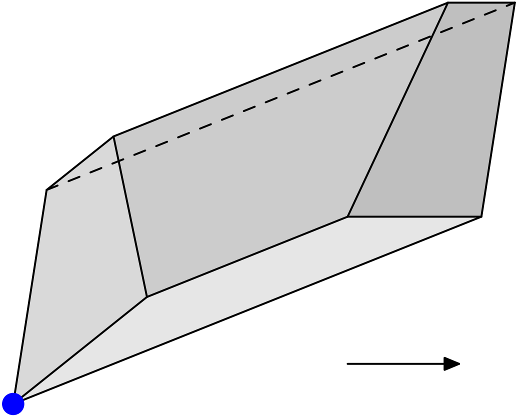


- The objective function defines an **orientation** of the **edges**.
- Many pivoting rules only rely on this orientation.
- RANDOM-EDGE: Perform a **random walk** until reaching the **sink** where all **edges** are incoming.

The RANDOM-FACET pivoting rule

- RANDOM-FACET: Introduced independently by Kalai [1992] and by Matoušek, Sharir and Welzl [1992].



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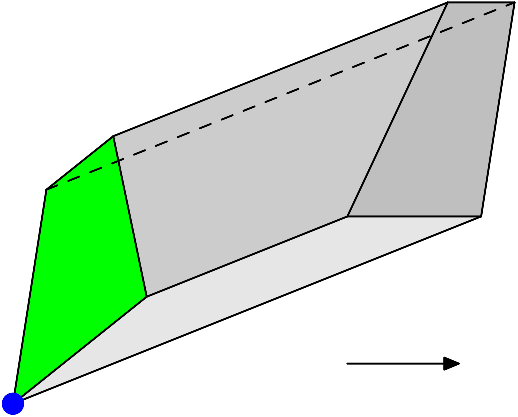

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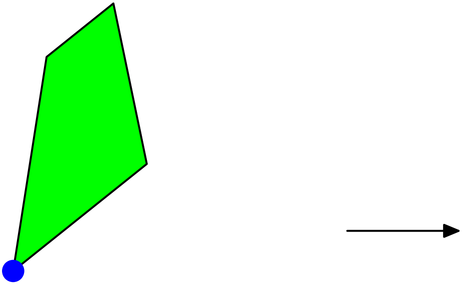

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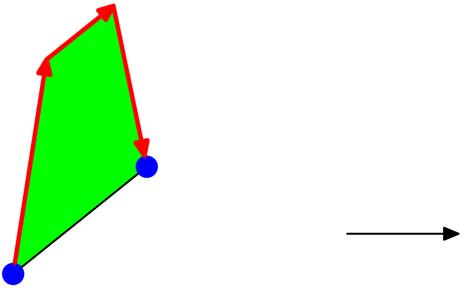

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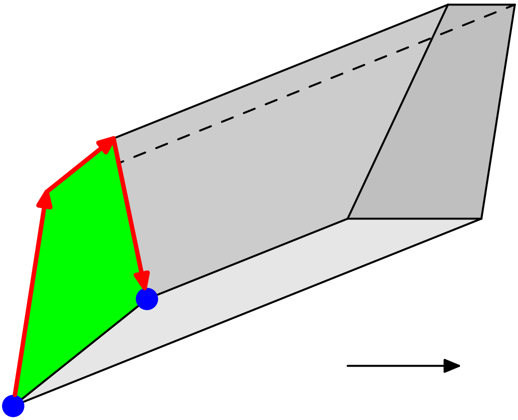

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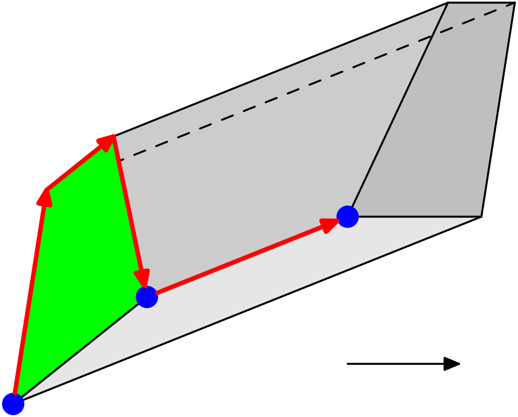

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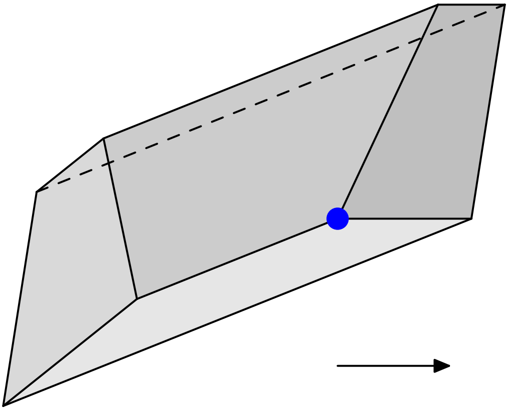

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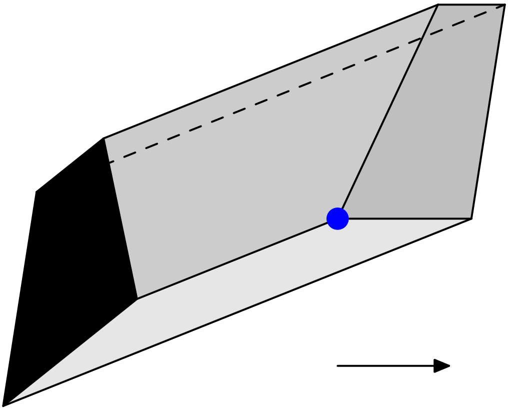
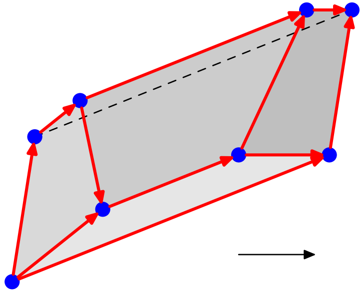


Properties of the orientation



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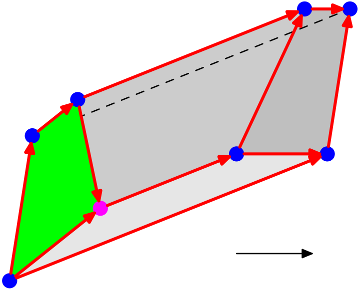
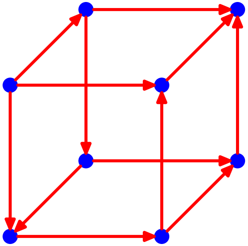


1. The graph is **acyclic**.
2. The **unique sink property**: In every **face** there is a **unique** **sink** (optimal **vertex** within the face).

Abstract cubes



- **Acyclic unique sink orientations** (AUSOs) (or **abstract objective functions**) are orientations that are
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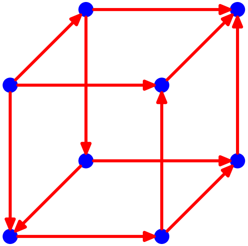

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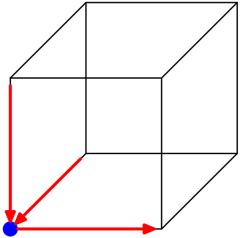
- AUSOs can be defined for arbitrary polytopes. We focus on the case where the underlying polytope is a **hypercube**.



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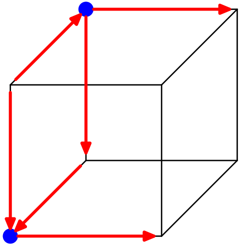

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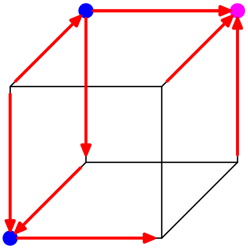

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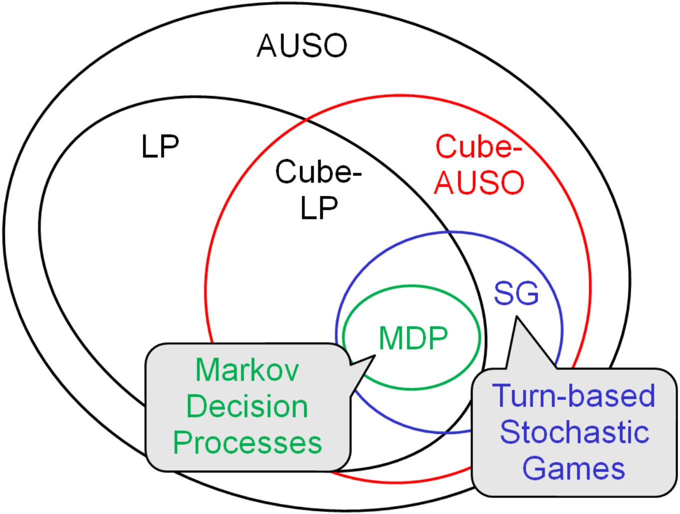
- **Goal:** Find the global **sink** with as few oracle calls as possible.

AUSOs and some applications



Results about Random-Facet

Bounds for the expected number of steps performed by Random-Facet on n-dimensional AUSOs with m facets.

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- Matoušek [1994]: $2^{\Omega(\sqrt{n})}$ for abstract cubes ($m = 2n$)
- Friedmann, Hansen, and Zwick [2011]: $2^{\tilde{\Omega}(\sqrt[3]{m})}$ for linear programs

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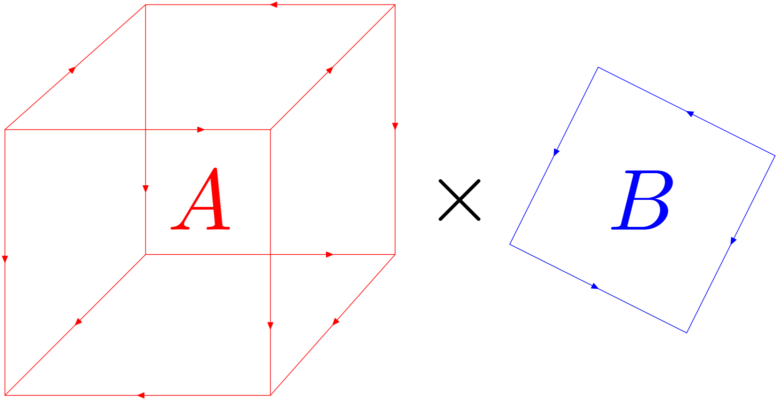
  - Improving the bound further requires significantly new ideas.

Open problem: Is RANDOM-EDGE subexponential?

Product of AUSOs [Schurr and Szabó, 2004]



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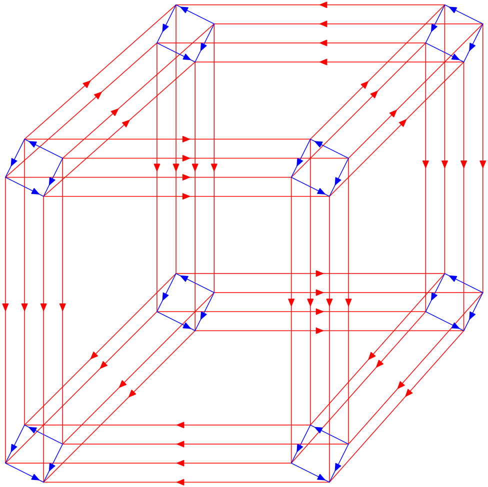

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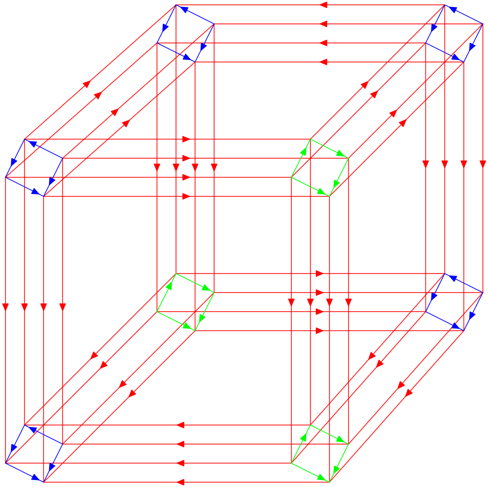
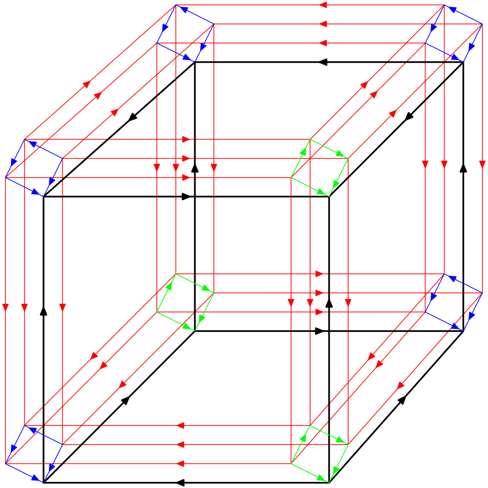


Hypersink replacement [Schurr and Szabó, 2004]



Matoušek and Szabó [2004]

- Let A be an AUSO for which Random-Edge performs T steps with high probability.
- **Goal**: Construct a slightly larger AUSO C for which Random-Edge performs $2T$ steps with high probability.

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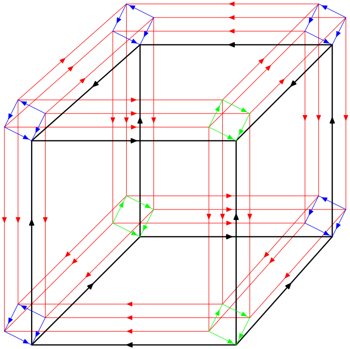
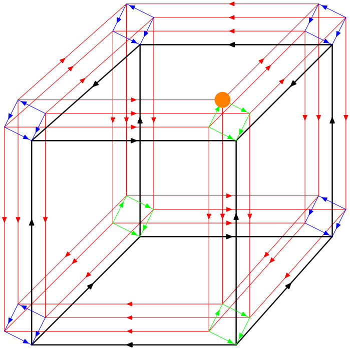


Random walk on product AUSO

- Every step in A brings us to a previously unvisited copy of B.
- Every copy of B has its coordinates randomly permuted.
- The hypersink is a randomly translated copy of A: This corresponds to starting from a uniformly random vertex of A.



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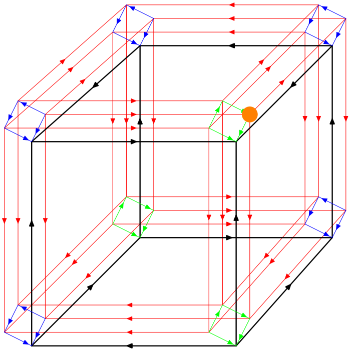

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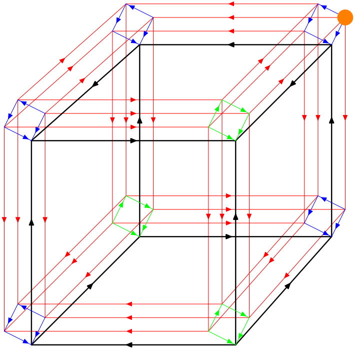

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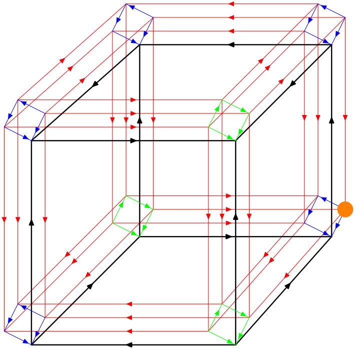

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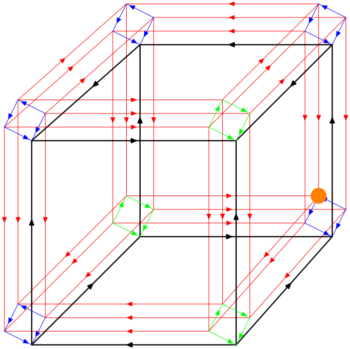

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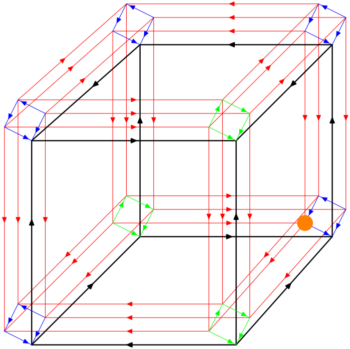

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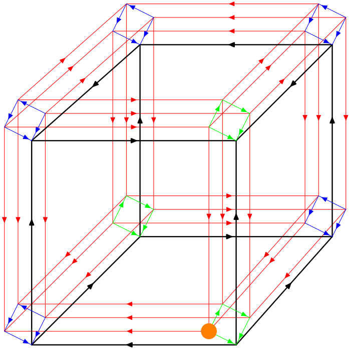

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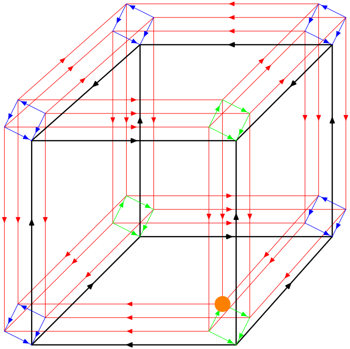

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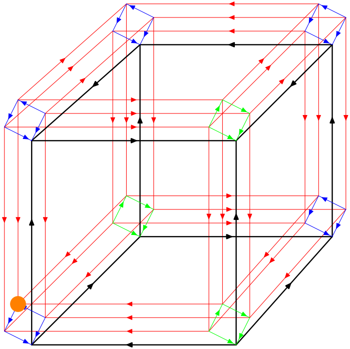

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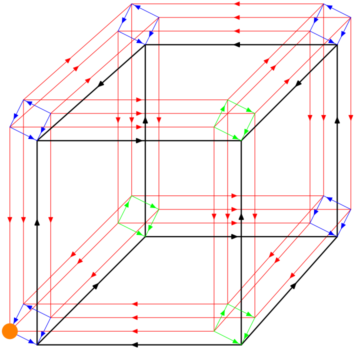

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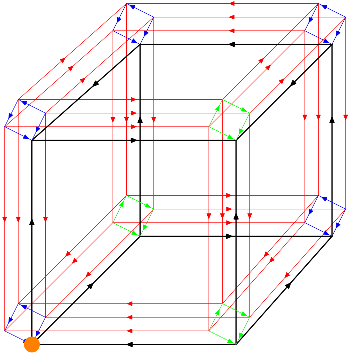

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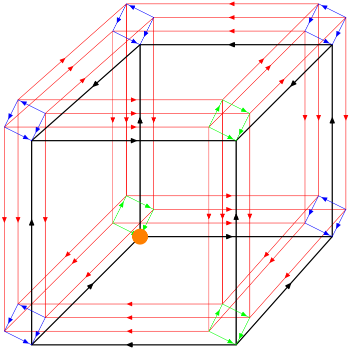

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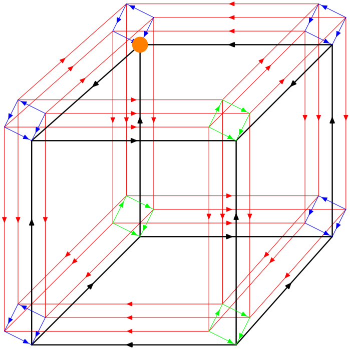

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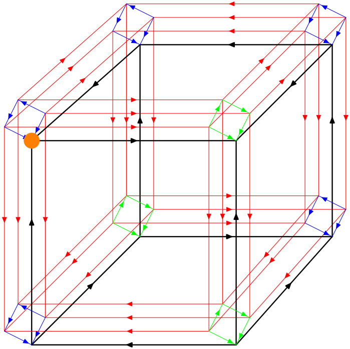

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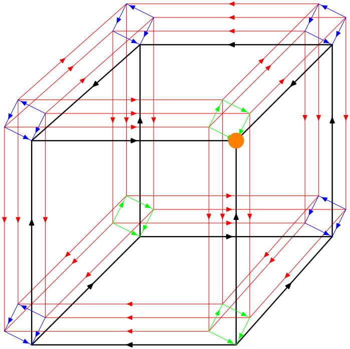
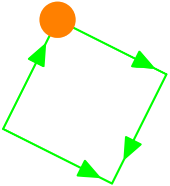


Random walk with reshuffles



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    - A larger k delays progress in B.

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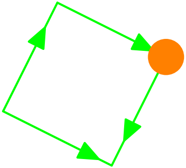

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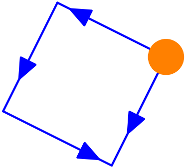

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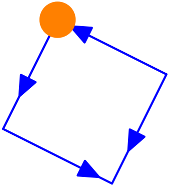

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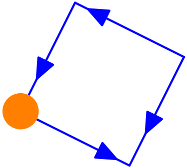

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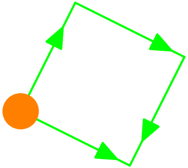

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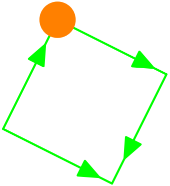

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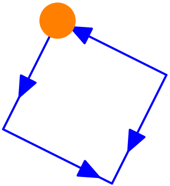

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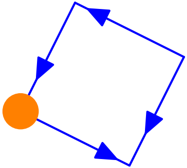

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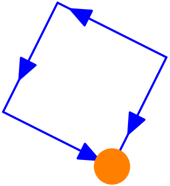

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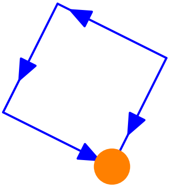

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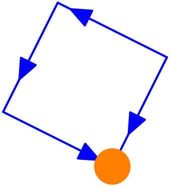

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- We simplify and improve their analysis by using only two copies (and therefore $k = 2$) of a **path AUSO**.

Path AUSOs

- Every vertex of a hypercube can be identified by a binary vector.
- **Path AUSO**: The i-th edge is outgoing iff the i-th coordinate is 0 and all previous coordinates are 1, or the i-th coordinate is 1 and some previous coordinate is 0.

In/**Out** : 11100110

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$$\textbf{In}/\textbf{Out}: \quad 11011101$$

- A reshuffle permutes the coordinates; the number of 0's and 1's remain unchanged.

In/**Out** : 11100110 $k = 5,\ j = 2$

- Suppose a vertex has k 1's, j of which are non-leading.
- Let $r \geq 2/(j+3)$ be the reshuffle probability for Random-Reshuffle$_2$ on B.

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- **Lemma**: For $8 \leq k \leq n-9$, the number of 1's increases with probability at most $5/12$.

$\textsc{Random-Reshuffle}_2$ on a path AUSO

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- **Lemma**: For $8 \leq k \leq n-9$, the number of 1's increases with probability at most $5/12$.
- The process can be analyzed as a **biased random walk** on $\{0, 1, \ldots, n-17\}$.

Choosing the size of B

By analyzing the biased random walk on $\{0, 1, \ldots, n\}$ we get:

Lemma

Let P_m be the m-dimensional path AUSO. There are constants $\alpha, \beta > 0$ such that the probability that $\textsc{Random-Reshuffle}_2$ *on P_m, starting from a random vertex, performs less than $2^{\alpha m}$ steps before reaching the sink is at most $2^{-\beta m}$.*

- We let $A_0 = P_m$ and $A_i = A_{i-1} \times_R^2 P_m$ for $i > 1$.
- We show that the probability that $\textsc{Random-Edge}$ performs less than 2^{ℓ} steps when started at a random vertex of A_ℓ, where $\ell < \alpha m$, is at most $4 \cdot 2^{\ell - \beta m}$.

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- Choosing $\ell = \Theta(m)$ gives a $2^{\Omega(\sqrt{n})}$ lower bound, where $n = \Theta(\ell m)$.

Improving the bound further

- We show that RANDOM-RESHUFFLE$_2$ on the m-dimensional path AUSO in **two steps** almost always increases the number of 1's with probability at most $O(1/\sqrt{m})$.
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- **Can we do better? No**:
  - Any m-dimensional AUSO has a path of length at most m to its sink from every vertex.
  - This is true for any choice of B in $A_i = A_{i-1} \times_R B$.
  - Random-Edge on A_ℓ follows this path in B with probability at least $1/n^m$, where m is the dimension of B and $n = m\ell$ is the dimension of A_ℓ.

$$\#steps \;\leq\; \min\{2^\ell, n^{n/\ell}\} poly(n)$$

  - It is impossible to get a better ℓ relative to m, regardless of the choice of B.

Concluding remarks

- We gave a $2^{\Omega(\sqrt{n \log n})}$ lower bound for Random-Edge on abstract cubes (AUSOs), showing that Random-Edge is slower than Random-Facet for this problem.
- **Open problem**: Is Random-Edge subexponential? Can the lower bound be further improved?
- **Open problem**: Improve the $2^{\Omega(\sqrt[4]{n})}$ lower bound for linear programming by Friedmann, Hansen, and Zwick [2011].
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Thank you for listening!