# Facial Reduction and Geometry on Conic Programming

*Masakazu Muramatsu*
UEC

*Bruno F. Lourenço, and Takashi Tsuchiya*
Seikei University GRIPS

Based on the papers of LMT:

1. *A structural geometrical analysis of weakly infeasible SDPs* (Journal of Operations Research Society of Japan 2016)
2. *Weak infeasibility in second order cone programming* (Optimization Letters 2015)
3. *Facial Reductions and Partial Polyhedrality* (Under Review)
4. (Under preparation)

*2016/8/13 WAO@Shinagawa*

NOTE:
This talk is a `Random Walk' within these works.

# Contents

1. Conic Programming (CP) and Duality
2. Feasibility Statuses of CP
3. Facial Reduction Algorithm
4. Distance to Polyhedrality and FRA-Poly
5. Cone Expansion and Feasibility Transition Theorems
6. Nasty Problems and FRA

dual

$$\theta_D = \sup\{by : c - A^T y \in K^*\} \leftrightarrow \theta_P = \inf\{cx : Ax = b,\ x \in K\}$$

primal/dual

dual/primal

# 1. CP and Duality

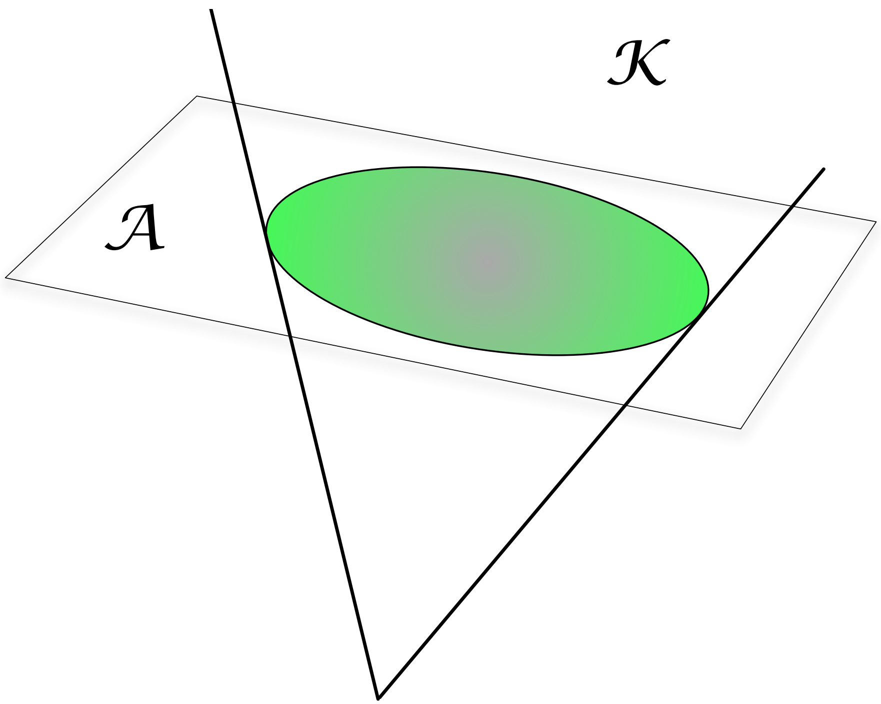
*dual*

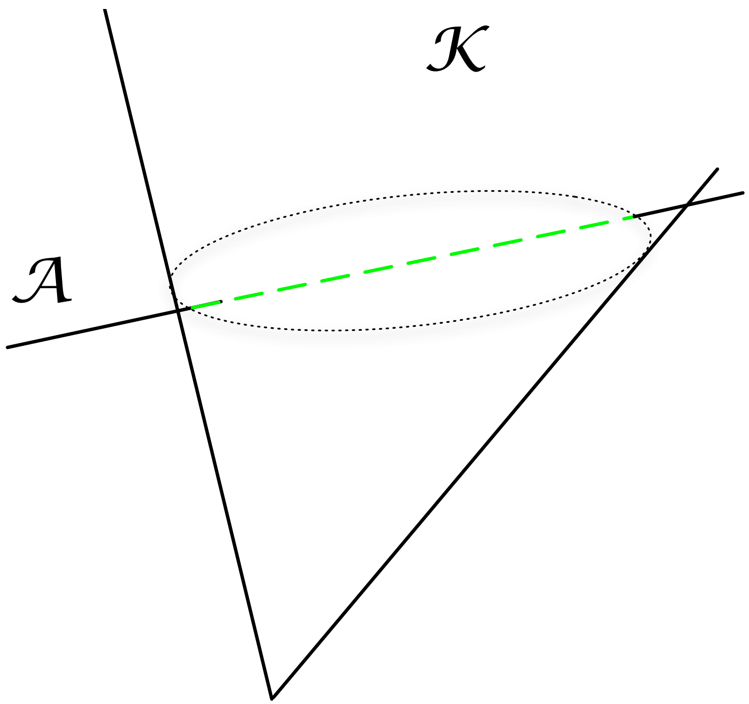
$$\theta_D = \sup\{by : c - A^T y \in K^*\} \leftrightarrow \theta_P = \inf\{cx : Ax = b,\ x \in K\}$$

$$\mathcal{A} = \{c - A^T y : y \in R^m\} \qquad \mathcal{A} = \{x : Ax = b\}$$

$\mathcal{A} \cap \mathcal{K}$ : Feasible Region

# Conic Programming



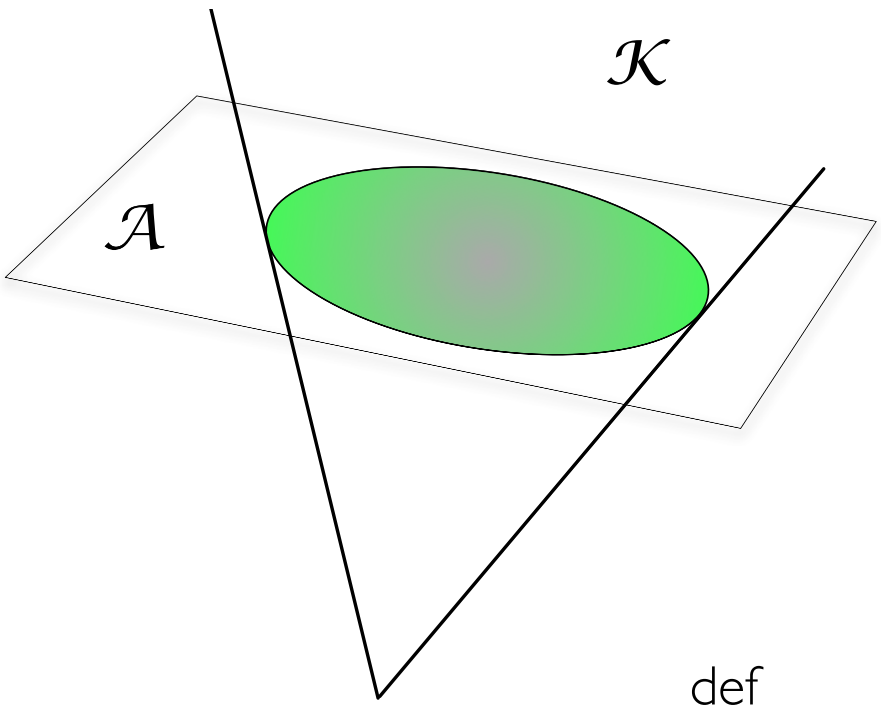

$\mathcal{K}$ : Closed Convex Cone

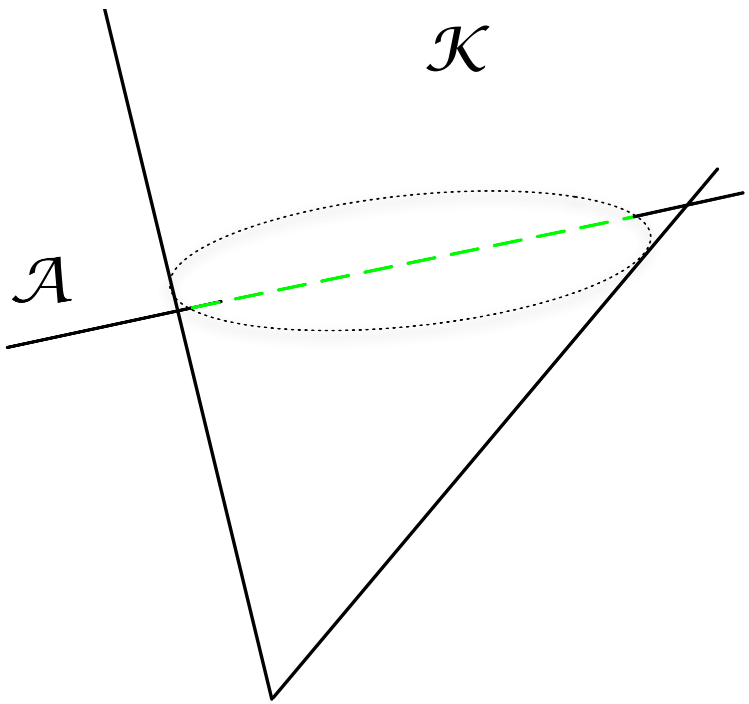
$\mathcal{A}$ : Affine Subspace

CP: ***Minimizing Linear Fn. over*** $\mathcal{A} \cap \mathcal{K}$

**Example**
**LP, SOCP, SDP,…**

# Conic Programming




$$x \in \mathcal{A} \cap \mathrm{rel}\mathcal{K} \overset{\text{def}}{\Leftrightarrow} x \text{ is an } \boxed{\text{interior feasible}} \text{ point.}$$

relative interior

# Duality Theorem and Nasty Cases

**Duality Theorem in CP**

If an ***interior feasible*** point exists for Primal

1. Zero Duality Gap
2. Dual has an optimal solution.

No interior feasible point

→ 1. Positive Duality Gap
2. Optimal value may not be attained

→ Hard to compute optimal value/solution

*Both Primal and Dual need interior feasible solutions to ensure existence of optimal solutions in both sides*

# Corruption of Computation

Waki, Nakata, and M (2012)

$\zeta_r$ : **Computed optimal values** by SeDuMi of SDP relaxations for Polynomial Optimization indexed by relaxation order $r = 2, 3, 4, \ldots$

| $r$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $\zeta_r$ | 0.0000 | 0.0000 | 0.0024 | 0.0761 | 0.6813 | 0.7862 |

**Fact**: The Optimal Value is **zero** for all r.

Significant Difference

One of the primal or dual does not have interior feasible solutions.

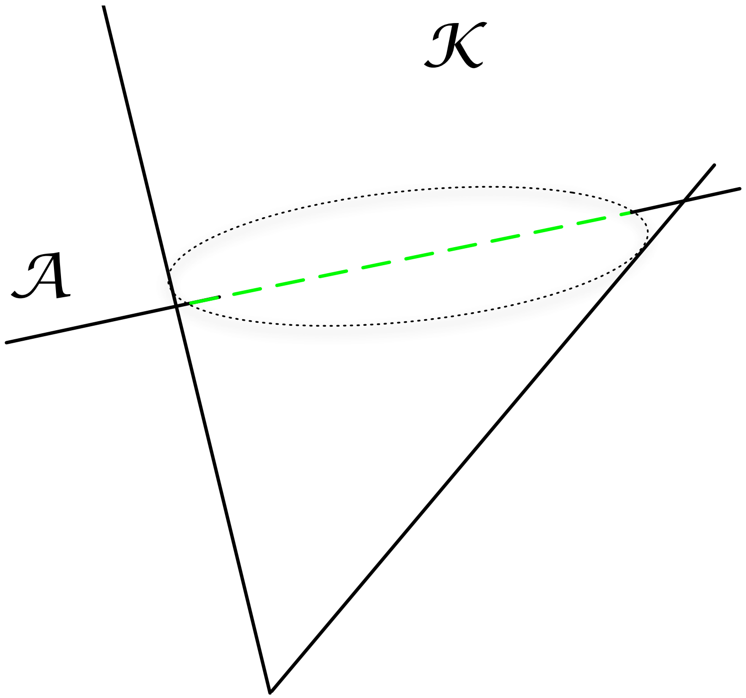
# 2. Feasibility Statuses of CP

$$\theta_D = \sup\{by : c - A^T y \in K^*\} \leftrightarrow \theta_P = \inf\{cx : Ax = b,\ x \in K\}$$

$$\mathcal{A} = \{c - A^T y : y \in R^m\} \qquad \mathcal{A} = \{x : Ax = b\}$$

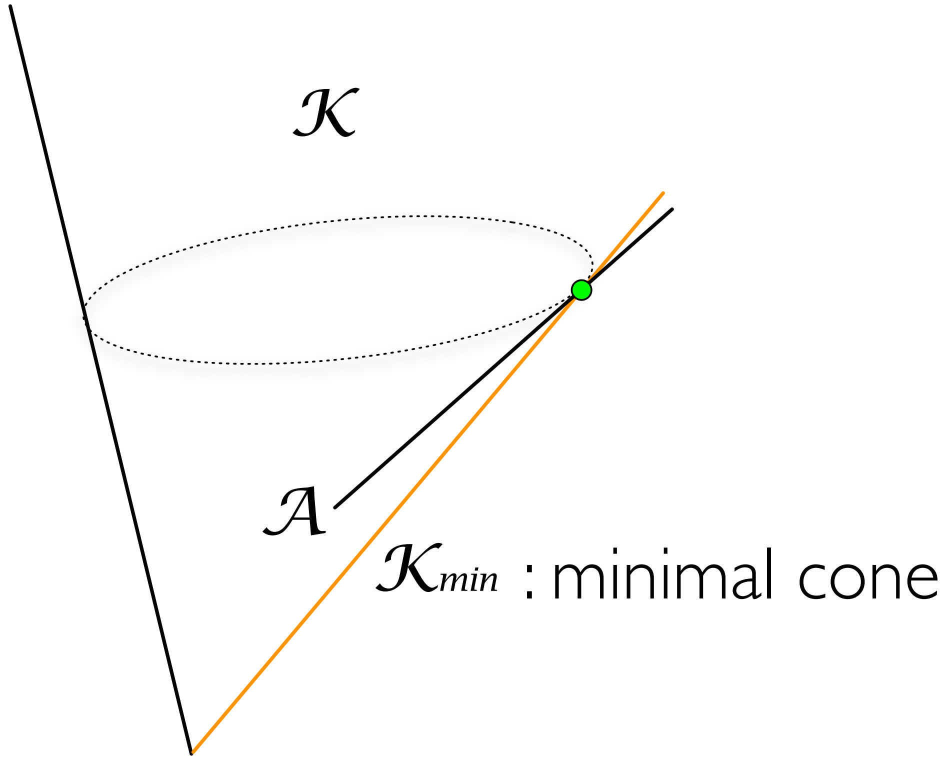
$\mathcal{A} \cap \mathcal{K}$ : Feasible Region

# Four Feasibility Statuses of Conic LP I.




$$\mathcal{A} \cap \mathrm{rel}\mathcal{K} \neq \emptyset$$

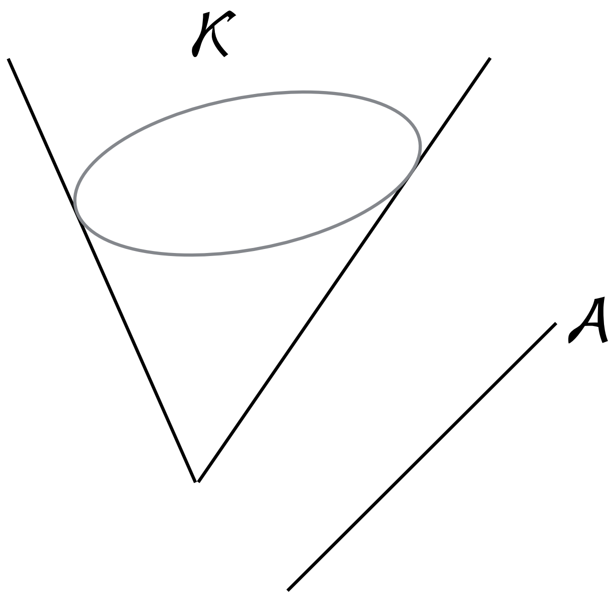
***Strongly Feasible***




$$\mathcal{A} \cap \mathcal{K} \neq \emptyset, \text{ but } \mathcal{A} \cap \mathrm{rel}\mathcal{K} = \emptyset$$

***Weakly Feasible***

# Four Feasibility Statuses of Conic LP II.




$$\mathrm{dist}(\mathcal{A}, \mathcal{K}) > 0$$

***Strongly Infeasible***

?

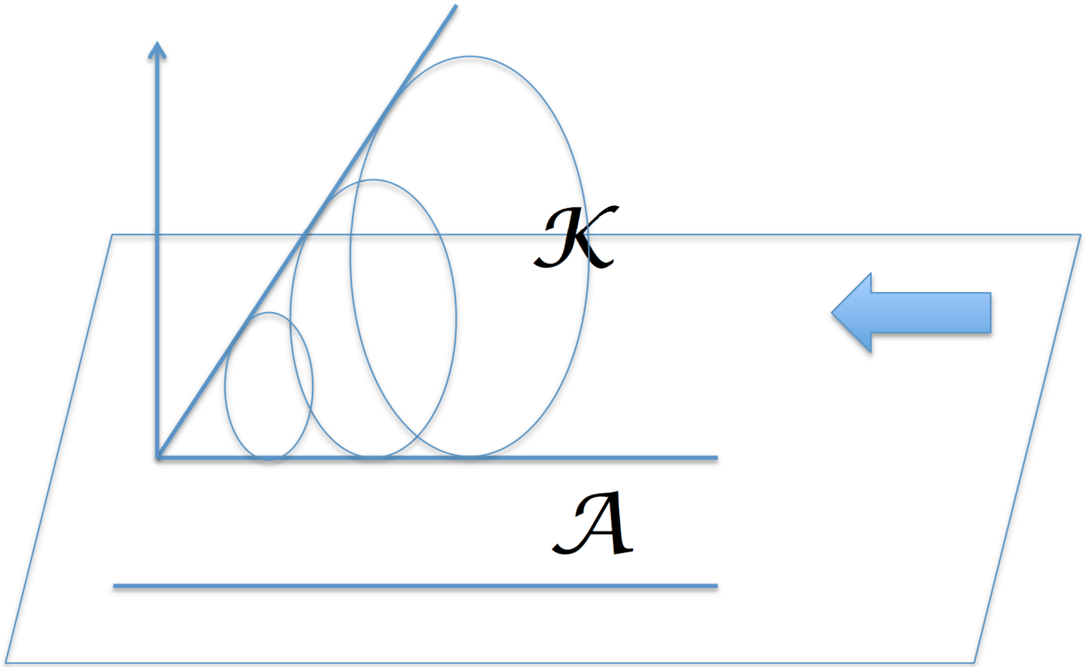
See the next slide…

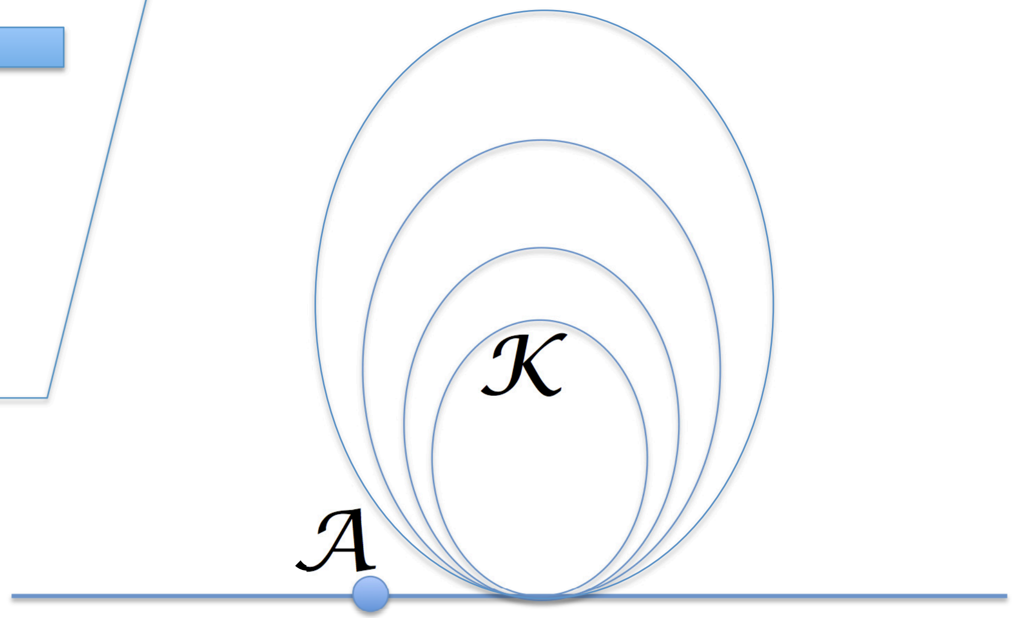
$$\mathrm{dist}(\mathcal{A}, \mathcal{K}) = 0 \text{ but } \mathcal{A} \cap \mathcal{K} = \emptyset$$

***Weakly Infeasible***

Impossible in LP

# Weakly Infeasible CP



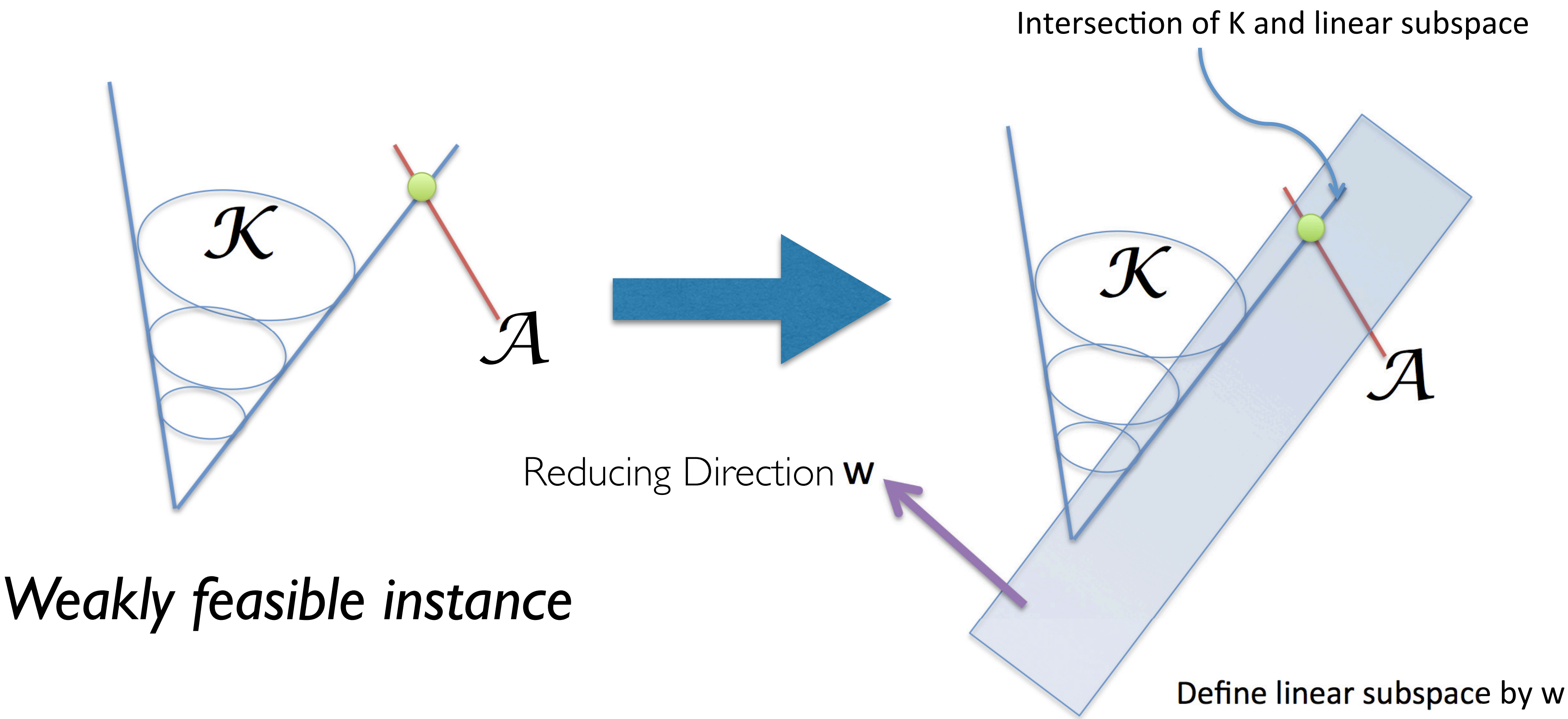




$\mathrm{dist}(\mathcal{A}, \mathcal{K}) = 0$ but $\mathcal{A} \cap \mathcal{K} = \emptyset$ : ***Weakly Infeasible***

# 3.Facial Reduction Algorithm

— How to obtain a well-behaved problem —

# Facial Reduction Algorithm(FRA)




Find w and take intersection of K and A.
Repeat this until the `minimal cone' is found

# FRA Details

$$\theta_D = \sup\{by : c - A^T y \in K^*\} \leftrightarrow \theta_P = \inf\{cx : Ax = b,\ x \in K\}$$

**FRA applied to** $\theta_D$ — Iterate the following steps

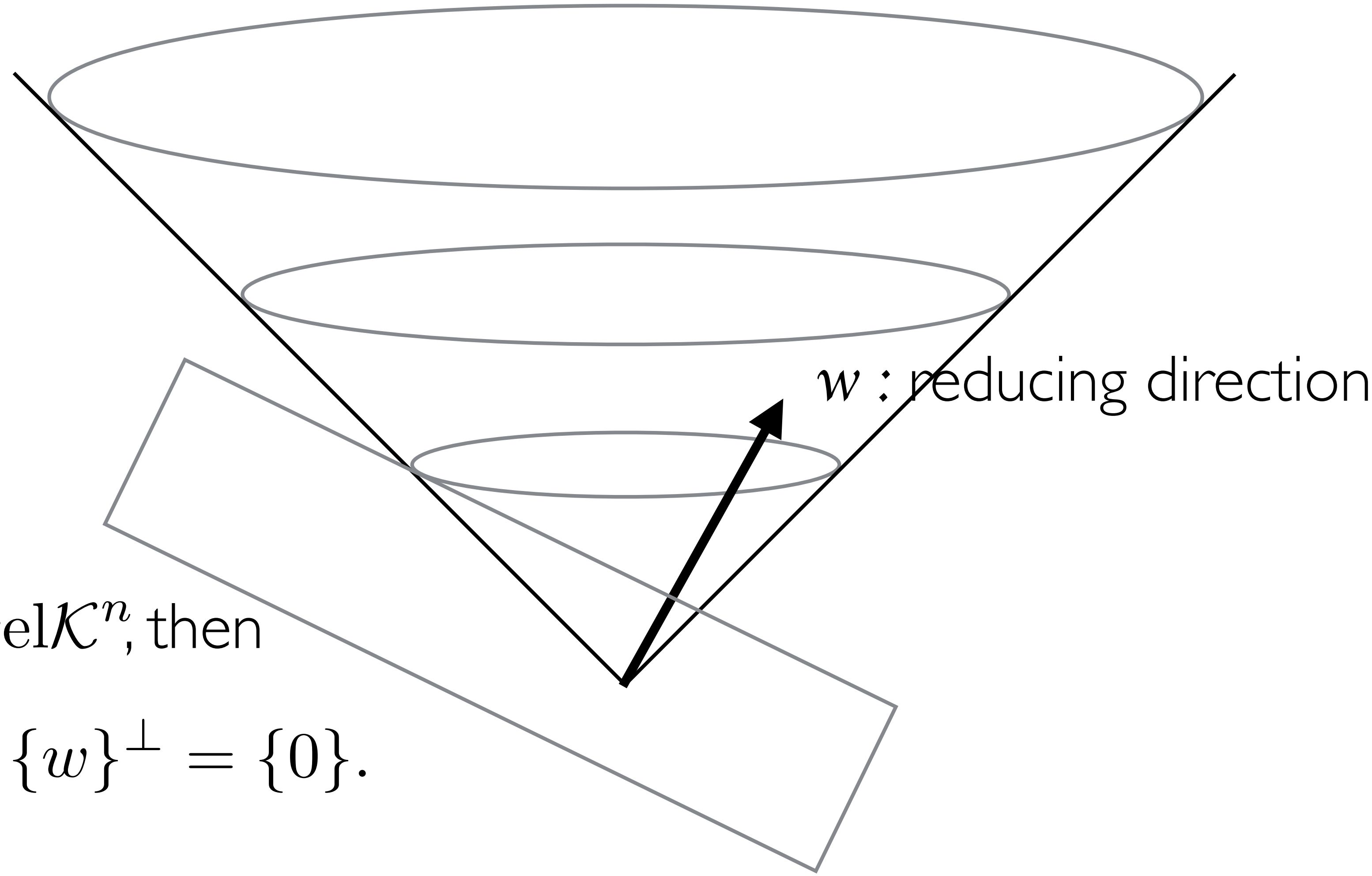
1. Find a ***reducing direction w***
2. Replace $\mathcal{K}^*$ in $\theta_D$ by $\mathcal{K}^* \cap \{w\}^\perp$

**Properties**

1. The iteration number is bounded by ***the length of the longest chain of faces of*** $\mathcal{K}^*$
2. When it stops, then we find either:
   - strongly feasible instance whose objective value is $\theta_D$
   - strongly infeasible instance, showing $\theta_D$ is infeasible

# Example: SOCP FRA

$$\mathcal{K}^n = \{(x_0, \tilde{x}) \in R^n : x_0 \geq \|\tilde{x}\|\}$$

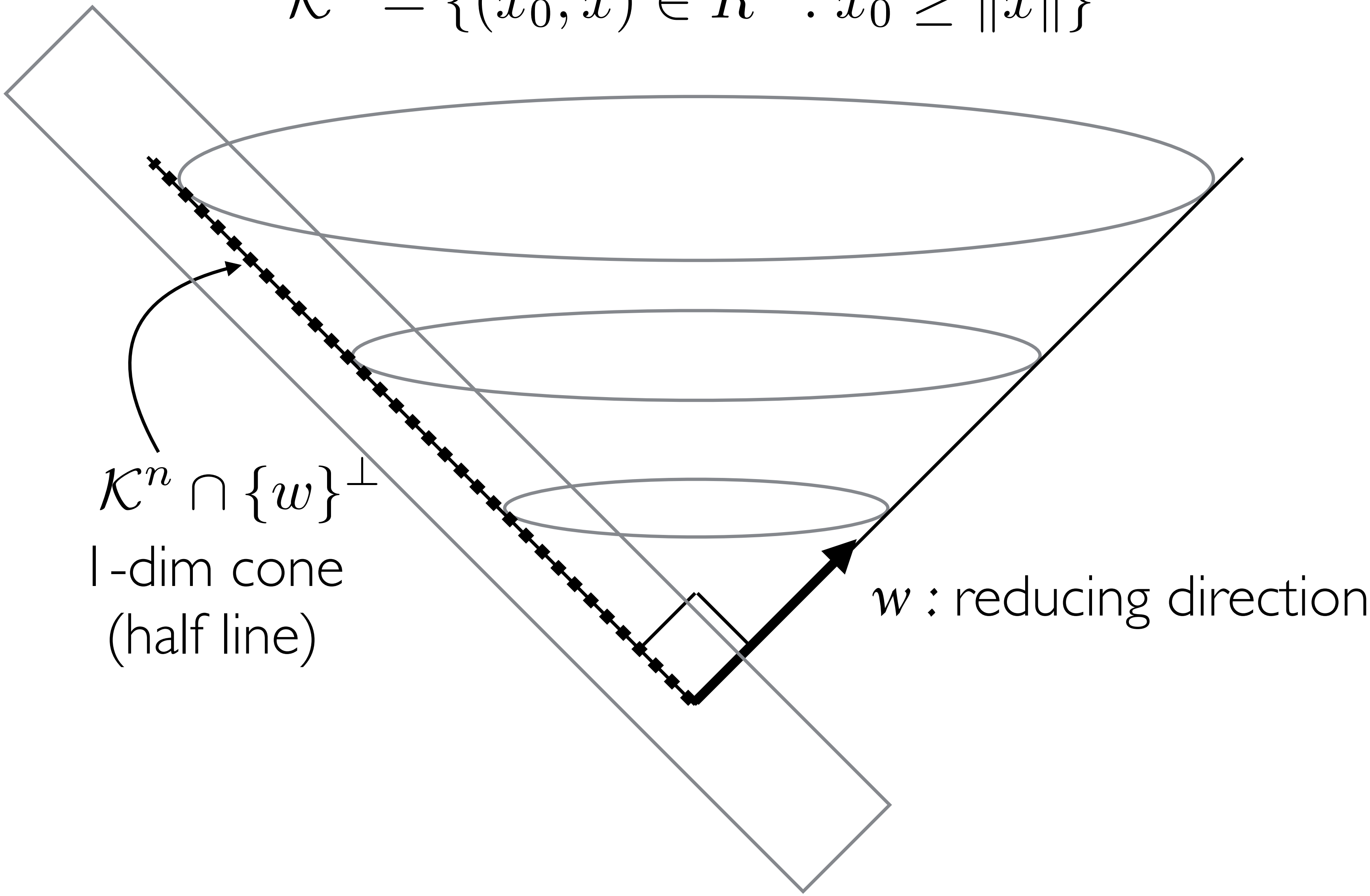



If $w \in \mathrm{rel}\mathcal{K}^n$, then

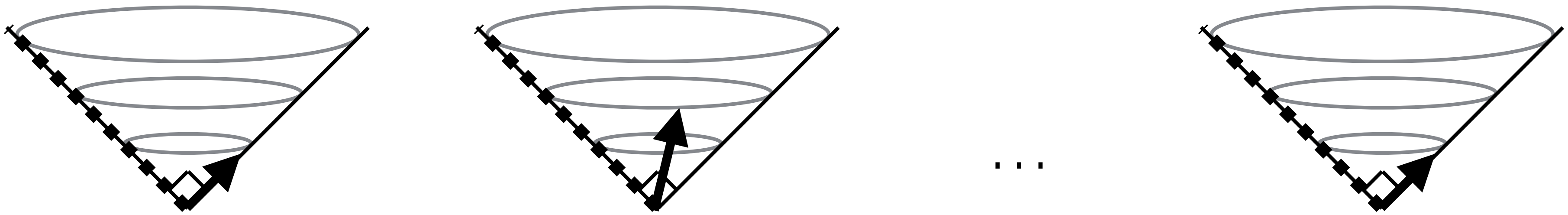
$$\mathcal{K}^n \cap \{w\}^{\perp} = \{0\}.$$

# Example: SOCP FRA

$$\mathcal{K}^n = \{(x_0, \tilde{x}) \in R^n : x_0 \geq \|\tilde{x}\|\}$$



# Example: SOCP FRA

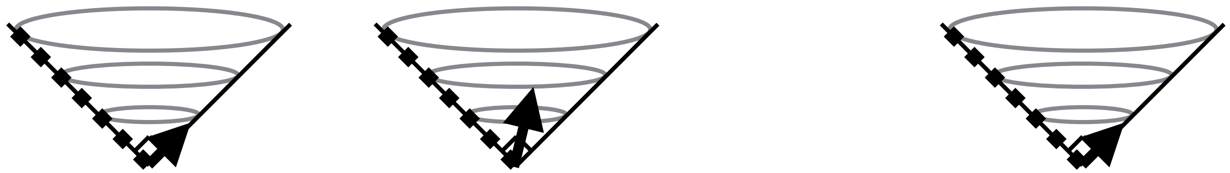


$$\mathcal{K}^{n_1} \times \mathcal{K}^{n_2} \times \cdots \times \mathcal{K}^{n_m}$$

1. One FRA iteration makes at least one SOC to polyhedral
2. At most m iteration is needed to obtain polyhedral cone
   - Enough to have a good property of duality

FRA-Poly

# 4. Distance to Polyhedrality and FRA-Poly



$$\mathcal{K}^{n_1} \quad \times \quad \mathcal{K}^{n_2} \quad \times \qquad \times \quad \mathcal{K}^{n_m}$$

# Distance to Polyhedrality

Observation:

- **No Nasty LPs** even if not strongly feasible
- If we reach a polyhedral cone, we are happy.

( if needed, just one more FRA is enough to obtain a strongly feasible LP)

1. $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_l = \mathcal{K}$
2. $\mathcal{F}_1$ is polyhedral
3. Others are non-polyhedral
4. Suppose that this chain is the longest one

→ $l - 1$ is called
***Distance to Polyhedrality***

# Partial Polyhedral Slater's (PPS) Condition

$\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$ where $\mathcal{K}_2$ is polyhedral.

CP satisfies PPS Condition

$$\Leftrightarrow \ \exists (x_1, x_2) \in \mathcal{A} \cap \mathcal{K} \text{ s.t. } x_1 \in \text{rel}\mathcal{K}_1$$

**Theorem.** If CP satisfies PPS Condition,

1. No duality gap
2. Dual is attained.

# FRA-Poly

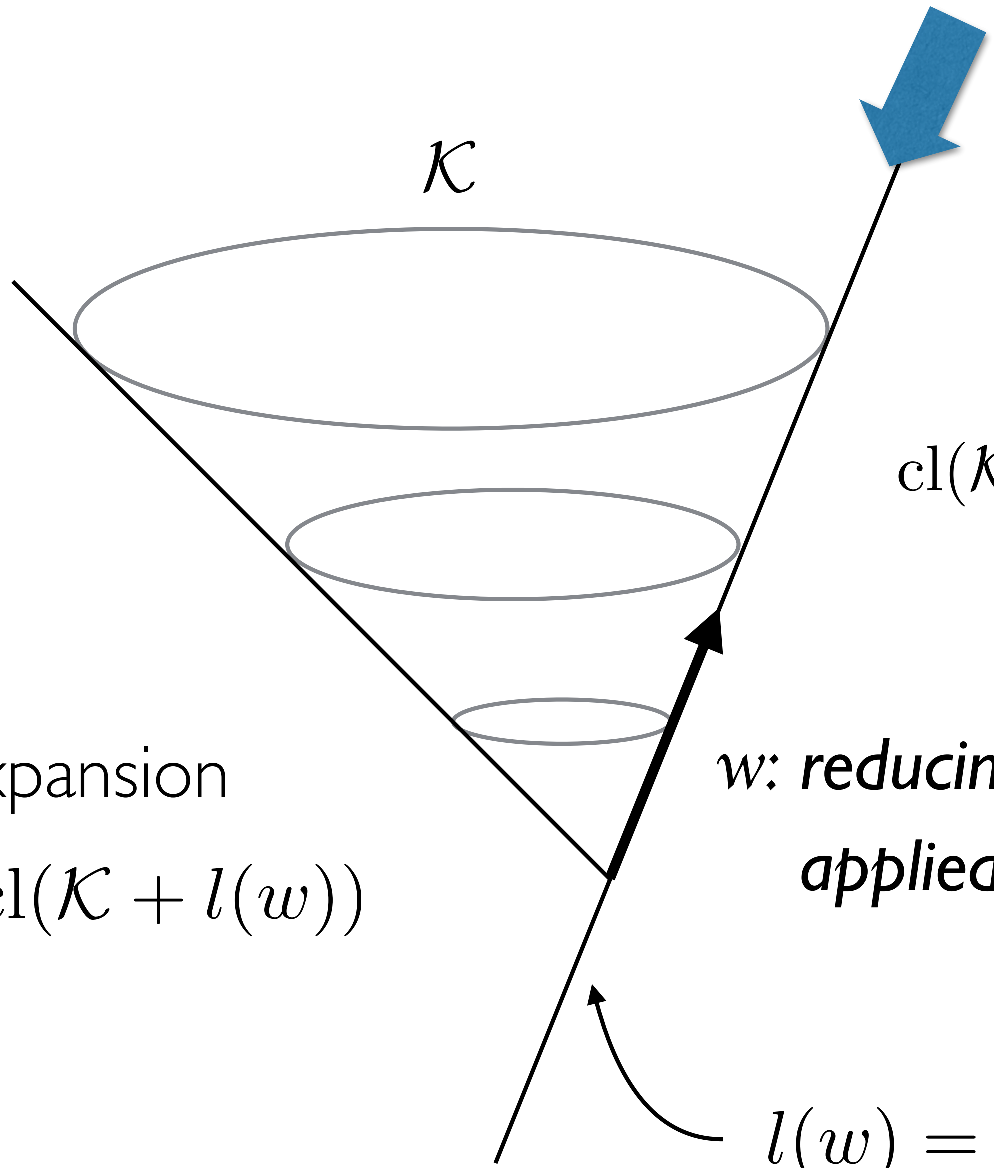
- We can construct FRA in such a way to reduce non-polyhedral cone (FRA-Poly).
- Distance-to-polyhedrality is an upper bound of the number of iterations of FRA-Poly.

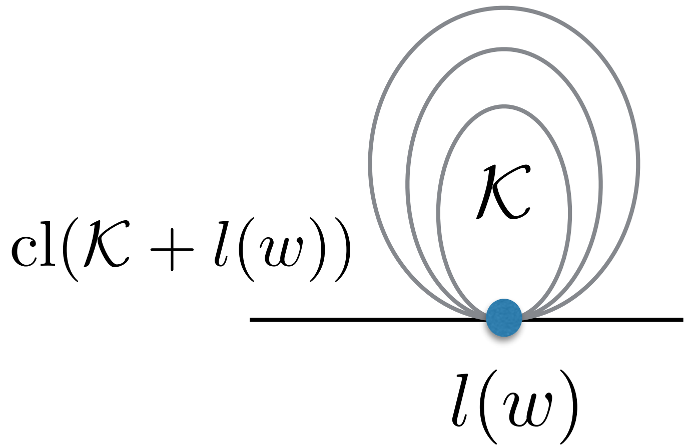
Upper bounds of FRA predicted by

| | the longest chain of Faces | Distance to Polyhedrality |
|---|---|---|
| SOC | 2 | 1 |
| PSD | n+1 | n |
| DNN | n(n+1)/2+1 | n |

# 5. Cone Expansion and Feasibility Transition Theorems

# Cone Expansion




Cone Expansion

$\mathcal{K} \mapsto \mathrm{cl}(\mathcal{K} + l(w))$

$w$: ***reducing direction of FRA applied to the dual program***

$l(w) = \{\lambda w : \lambda \in R\}$

linear subspace spanned by w

# Cone Expansion (CE)

$$\theta_D = \sup\{by : c - A^T y \in K^*\} \leftrightarrow \theta_P = \inf\{cx : Ax = b,\ x \in K\}$$

FRA

Project the primal cone

dual $\longleftrightarrow$

CE

Expand the dual cone

$$\theta'_D = \sup\{by : c - A^T y \in K^* \cap w^\perp\}$$
$$\leftrightarrow \theta'_P = \inf\{cx : Ax = b,\ x \in \mathrm{cl}(K + l(w))\}$$

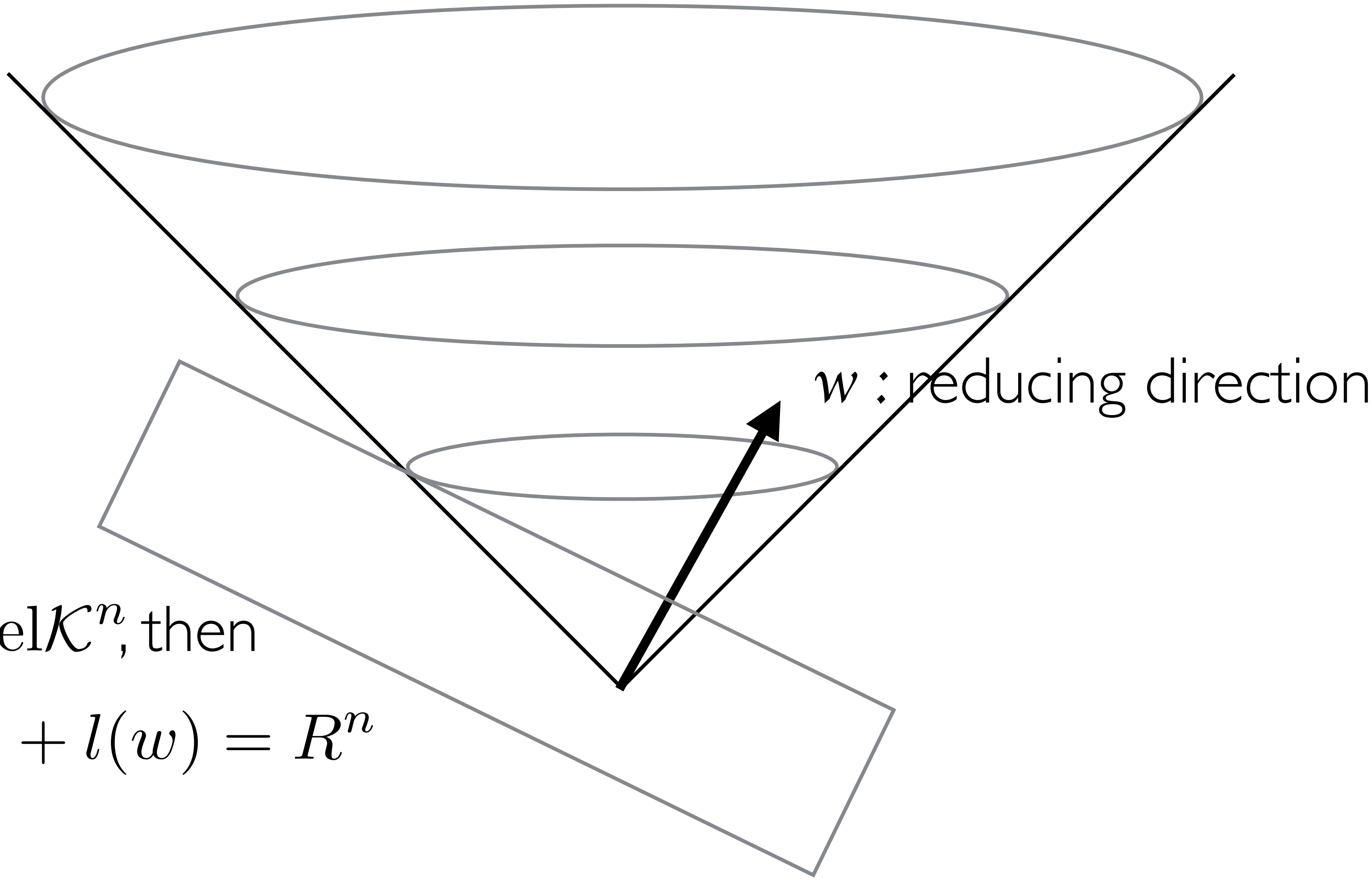
(Specially chosen w corresponds to

Luo, Sturm, Zhang; Waki, M)

# Example: SOCP CE 1.

Dual of SOC is SOC
(Self Dual)

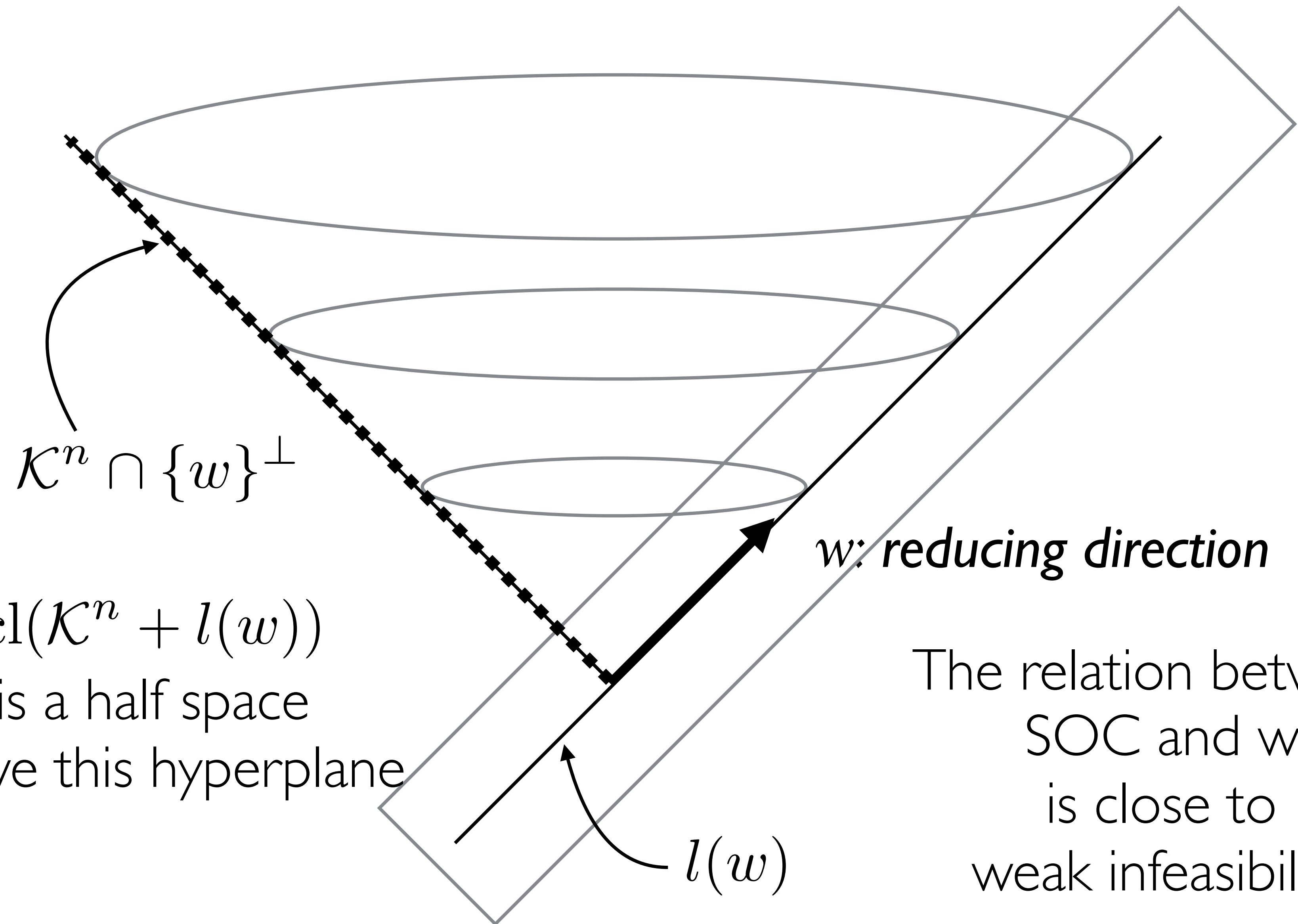
SOC is full-dimensional




If $w \in \mathrm{rel}\mathcal{K}^n$, then

$$\mathcal{K}^n + l(w) = R^n$$

# Example: SOCP CE 2.




$\mathrm{cl}(\mathcal{K}^n + l(w))$
is a half space
above this hyperplane

The relation between
SOC and w
is close to
weak infeasibility

# FRA, CE and Feasibility

$$\text{Primal Problem} \quad \theta_D \xrightarrow{\text{FRA}} \theta'_D$$

$$\text{Dual Problem} \quad \theta_P \xrightarrow{\text{CE}} \theta'_P$$

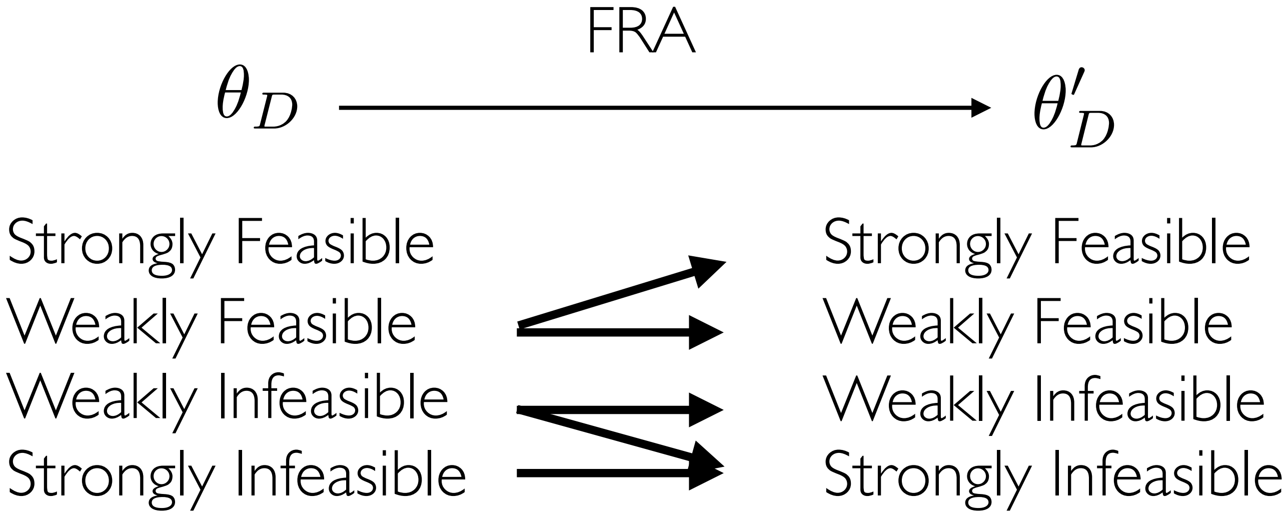
1. FRA **does not change** the feasible region
2. Feasible region of $\theta'_P$ could be ***larger*** than that of $\theta_P$

$$\longrightarrow \theta'_D = \theta_D,\ \theta'_P \le \theta_P$$

$$\overbrace{\theta_D = \theta_D^0 = \theta_D^1 = \ldots = \theta_D^p}^{\text{FRA}}$$

$$\underbrace{\theta_P = \theta_P^0 \le \theta_P^1 \le \ldots \le \theta_P^p}_{\text{CE}}$$

# Feasibility Transition by FRA




As long as the problem is in weak status,
we can apply FRA.

Final status: strongly feasible or infeasible instance

# Feasibility Transition by CE

LTM

$$\theta_P \xrightarrow{\text{CE}} \theta'_P$$

| | | |
|---|---|---|
| Strongly Feasible | → | Strongly Feasible |
| Weakly Feasible | → | Weakly Feasible |
| Weakly Infeasible | → | Weakly Infeasible |
| Strongly Infeasible | → | Strongly Infeasible |

As long as the problem weakly infeasible,
we can apply CE.

→ Final status: Feasible, or strongly infeasible instance.

# Strongly Feasible but Non-attained problem

$$\theta_D = \sup\{by : c - A^T y \in K^*\} \leftrightarrow \theta_P = \inf\{cx : Ax = b,\ x \in K\}$$

- Suppose that $\theta_D$ is <u>strongly feasible but not attained</u>.
- Apply FRA to $\theta_P$ to obtain the final problem $\theta_P^p$

(Equivalently, Apply CE to $\theta_D$ to obtain $\theta_D^p$)

Strongly feasible

$$\begin{array}{ccccccccc} \theta_P & = & \theta_P^0 & \stackrel{FRA}{\rightarrow} & \theta_P^1 & \stackrel{FRA}{\rightarrow} & \cdots & \stackrel{FRA}{\rightarrow} & \theta_P^p \\ \theta_D & = & \theta_D^0 & \stackrel{CE}{\rightarrow} & \theta_D^1 & \stackrel{CE}{\rightarrow} & \cdots & \stackrel{CE}{\rightarrow} & \boxed{\theta_D^p} \end{array}$$

Strongly feasible but not attained

Strongly feasible and attained

# 6. Nasty Problems and FRA

# Computing an approximate optimal solution

**Aim. Given $\epsilon > 0$ find an feasible solution of $\theta_D$ whose obj. value $> \theta_D - \epsilon$**

- Since $\theta_D^p$ is strongly feasible by FTT for CE,
  $$\exists \hat{y},\ c - A^T \hat{y} \in \mathrm{rel}\hat{K}^*$$
- Let $y^*$ be an optimal solution of $\theta_D^p$

***the cone of*** $\theta_D^p$

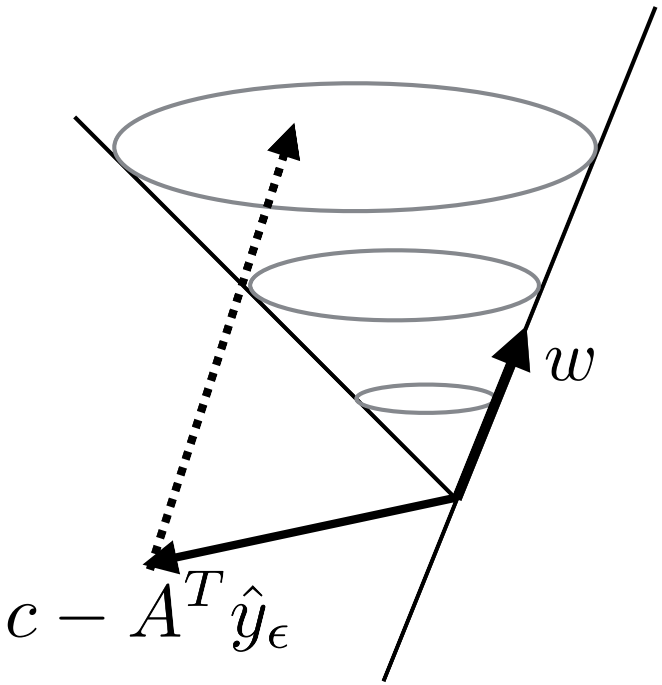
It is easy to compute a feasible solution of $\theta_D^p$ whose obj. value $> \theta_D - \epsilon$ using the above.

Let $\hat{y}_\epsilon$ be such a solution.

# How to compute an approximate optimal solution

- Let $w_1, \ldots, w_p$ be the reducing directions.
- There exists positive numbers $\alpha_1, \ldots, \alpha_p$ such that

$$c - A^T \hat{y}_\epsilon + \sum_{i=1}^{p} \alpha_i w_i \in \mathcal{K}.$$




Cone Expansion

$$\mathcal{K} \mapsto \mathrm{cl}(\mathcal{K} + l(w))$$

# Properties of Reducing Direction

Let $w_1, \ldots, w_p \in \mathcal{K}$ be reducing directions of FRA applied to $\theta_D$. $(p \leq n)$

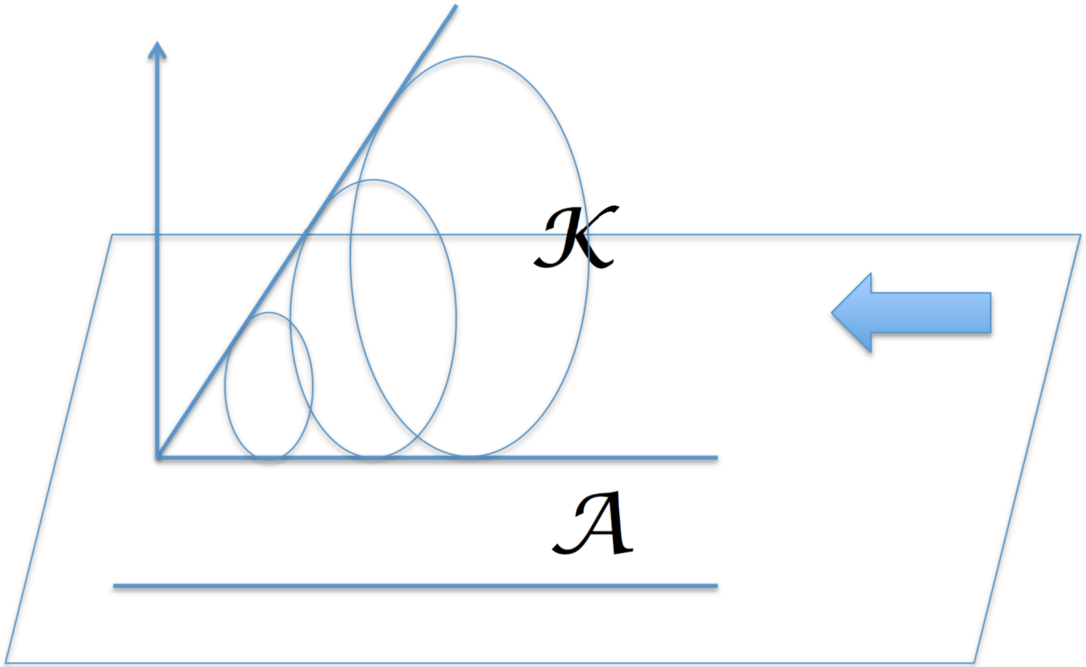
*If* $\theta_P$ *is weakly infeasible*, then

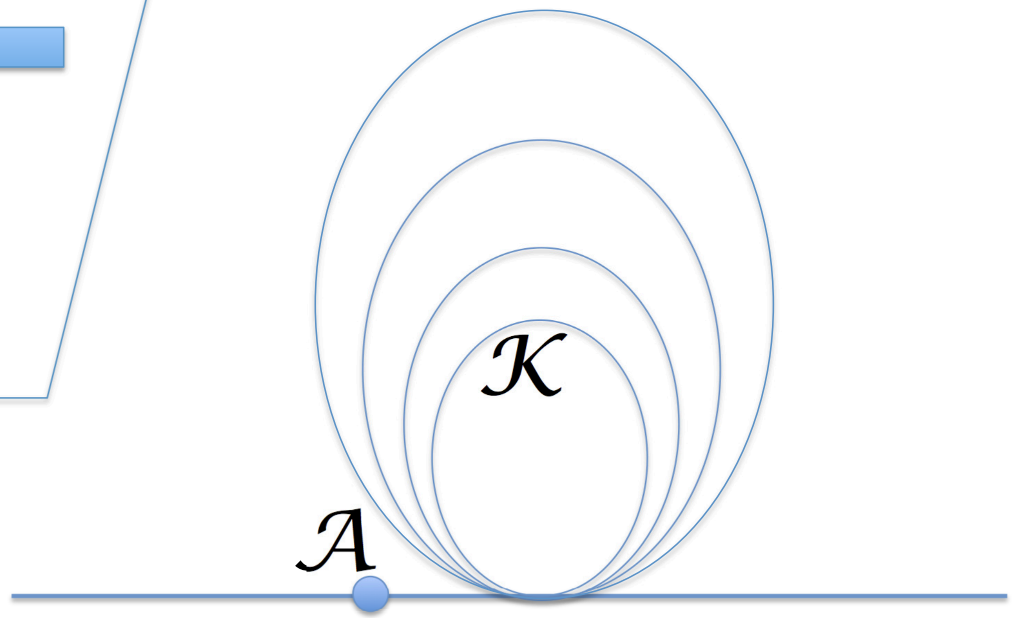
$$(c + \mathrm{span}(\underset{\substack{\rotatebox{90}{$\supseteq$} \\ \mathcal{A}}}{\underline{w_1, \ldots, w_p}})) \cap \mathcal{K} \text{ is also weakly infeasible.}$$

directions approaching the cone

In case of SOCP or SDP, given a positive number $\epsilon$, we can explicitly compute a point on $\mathcal{A}$ whose distance from the cone is less than $\epsilon$

# Misleading Picture of Weak Infeasibility




We need p>0 directions to approach K in general.
These directions are \`reducing directions'.

# Thank you
# and
# Happy Birthday, Mizuno Sensei

The papers by Lourenço, M. and Tsuchiya:

1. A structural geometrical analysis of weakly infeasible SDPs (Journal of Operations Research Society of Japan 2016)
2. Weak infeasibility in second order cone programming (Optimization Letters 2015)
3. Facial Reductions and Partial Polyhedrality (Under Review)
4. (Under preparation)

# Example: FRA and CE on SDP

$$w = \begin{pmatrix} O & O \\ O & \oplus \end{pmatrix}$$ : reducing direction

FRA

CE

$$S^n_+ \cap \{w\}^\perp = \begin{pmatrix} \oplus & O \\ O & O \end{pmatrix}$$

$$\mathrm{cl}(S^n_+ + l(w)) = \begin{pmatrix} \oplus & * \\ * & * \end{pmatrix}$$

NOTE: The resulting problems are again SDP