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Note on the Relationship Between Partnership Formation Problem and Assignment Game

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Abstract

We discuss the partnership formation problem introduced by Talman and Yang (2011), which is a generalization the classical assignment game. We show that the partnership formation problem can be reduced to the assignment game in some sense; more precisely, we show that using an equilibrium in a certain assignment game, we can find an equilibrium in the partnership formation problem (if it exists). Based on this, we devise an algorithm to compute an equilibrium of a partnership formation problem. We also show that our algorithm can be seen as a generalization of the one by Andersson et al. (2014a).

Keywords: Partnership formation; Equilibrium; Assignment game; Adjustment process.

1. Introduction

In this paper we consider the partnership formation problem (Andersson et al., 2014a; Talman and Yang, 2011); this problem is also called the one-sided assignment problem (Klaus and Nichifor, 2010), and the roommate problem/game with transferable utility (Chiappori et al., 2012; Eriksson and Karlander, 2001). In the partnership formation problem, there is a group of agents, and each agents either acts alone or seeks a partner for cooperation. If an agent acts alone, then she generates a value for herself, and if an agent work with a partner, then the agent and her partner generate a joint value, which is shared by them in an appropriate way. The goal of the partnership formation problem is to find an equilibrium, where

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no agent has incentive to change her partner, to break up an existing partnership to become alone, or to form a new partnership. Typical instances of the partnership formation problem can be found in the professional tennis tournament, pair programming in software development, etc. (see, e.g., Andersson et al. (2014a); Eriksson and Karlander (2001); Talman and Yang (2011)). Similar but different models of the partnership formation are also discussed in Chiappori et al. (2012); Alkan and Tuncay (2013).

The partnership formation problem is deeply related to the classical assignment game (Koopmans and Beckmann, 1957; Shapley and Shubik, 1971). The assignment game can be regarded as a special case of the partnership formation problem, where the set of agents is partitioned into two groups, the one corresponding to buyers (or firms) and the other to sellers (or workers), and any two agents in the same group cannot be a pair. That is, every assignment game can be reduced to a partnership formation problem (see Talman and Yang (2011, Theorem 3); see also Chiappori et al. (2012)). It is known that an equilibrium always exists in the assignment game (Koopmans and Beckmann, 1957; Shapley and Shubik, 1971), and price adjustment processes for findings an equilibrium are proposed by Crawford and Knoer (1981) and Demange et al. (1986).

In contrast to the assignment game, the partnership formation problem may not have an equilibrium (see, e.g., Chiappori et al. (2012); Talman and Yang (2011)). Various sufficient (and necessary) conditions for the existence of equilibrium are provided by Eriksson and Karlander (2001) and Talman and Yang (2011). While the existing price adjustment processes for the assignment game cannot be applied to the partnership formation problem, a novel price adjustment process for the partnership formation problem is proposed by Andersson et al. (2014a), which can always either find an equilibrium or disprove the existence of an equilibrium in a finite number of iterations. In the price adjustment process by Andersson et al. (2014a), a certain payoff vector is computed in a way similar to the one by Demange et al. (1986), and using the payoff vector the existence of an equilibrium is determined with the aid of a matching algorithm.

As mentioned above, the assignment game can be reduced to the partnership formation problem. The main aim of this paper is to show that the converse is also true in some sense; more precisely, we show that using an equilibrium in a certain assignment game, we can find an equilibrium in the partnership formation problem (if it exists). For this, we define an assignment game associated with a given partnership formation problem and show the relationship with the partnership formation problem. We first prove that an equilibrium of the partnership formation problem corresponds to a "symmetric" equilibrium of the associated assignment game (see Theorem 3.4). Then, it is shown that using an equilibrium payoff in the associated assignment game, the problem of finding an equilibrium in the partnership formation problem can be reduced to the problem of finding a matching among the agents such that each agent is matched to one of her favorite agents (see Theorem 3.5). Base on the theorems, we present an algorithm for computing an equilibrium of the partnership formation problem.

It is observed that our algorithm is similar to the price adjustment process by Andersson et al. (2014a). Indeed, the starting point of our current research is to understand the behavior of the price adjustment process by Andersson et al. (2014a). We discuss its connection to our algorithm, and show that a special implementation of our algorithm exactly coincides with the price adjustment process by Andersson et al. (2014a).

Finally, it should be mentioned that Chiappori et al. (2012) also use an assignment game associated with the partnership formation problem, which is the same as ours, but for a different purpose. Chiappori et al. (2012) mainly consider the situation where each $i \in N$ is regarded as a type of agents and there are multiple agents in each type, i.e., their model of the partnership formation is different from ours. By using the connection between the partnership formation problem and the associated assignment game, Chiappori et al. (2012) show (i) the existence of an equilibrium matching in the case where there are even number of agents in each type, and (ii) the existence of a near-equilibrium matching (they call it a quasi-stable matching) in the case where the number of agents in each type is sufficiently large. In contrast, we focus on the case where there is a single agent in each type, and demonstrate how the associated assignment game is useful in computing an equilibrium in the partnership formation problem.

2. Preliminaries

We review the definitions and fundamental properties of the partnership formation problem and the assignment game.

2.1. Partnership Formation Problem

We denote by $N = \{1, 2, \ldots, n\}$ a set of agents, where n is a positive integer. A *partnership formation problem* is represented by the pair (N, v) of a set of agents N and values $v = (v_{ij} \mid i, j \in N)$ satisfying $v_{ij} = v_{ji}$. For distinct $i, j \in N$, $v_{ij} = v_{ji} \in \mathbb{R}$ denotes the (joint) value generated by the agents i and j. For each $i \in N$, $v_{ii} \in \mathbb{R}$ denotes the value generated by the single agent i; we assume, without loss of generality, that $v_{ii} = 0$.

The concept of equilibrium in the partnership formation problem is defined as follows. A *matching* is a function $\mu : N \to N$ such that for $i, j \in N$, we have $\mu(i) = j$ if and only if $\mu(j) = i$. That is, a matching corresponds to a partition of N into pairs of agents and/or single agents. A vector $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^N$ is called a *payoff*. A pair of a matching μ and a payoff p is called an *equilibrium* of the partnership formation problem (N, v) if the following conditions hold:

$$\left.\begin{array}{ll} p_i + p_j \geq v_{ij} & (\forall i, j \in N), \\ p_i + p_j = v_{ij} & (\text{if } \mu(i) = j \text{ and } i \neq j), \\ p_i \geq 0 & (\forall i \in N), \\ p_i = 0 & (\text{if } \mu(i) = i). \end{array}\right\} \tag{2.1}$$

An equilibrium of a partnership formation problem may not exist (see, e.g., Chiappori et al. (2012); Talman and Yang (2011)). For example, it is not difficult to see that the partnership formation problem with $n = 3$, $v_{ij} = 1$ for distinct i, j, and $v_{ii} = 0$ for $i = 1, 2, 3$ has no equilibrium.

A matching (resp., a payoff) in an equilibrium is called an *equilibrium matching* (resp., an *equilibrium payoff*). It should be noted that if $v_{ij} < 0$ for some distinct agents $i, j \in N$, then the agents i, j cannot be a pair in any equilibrium matching.

Below we review some fundamental facts on equilibria of the partnership formation problem. For an agent $i \in N$ and a payoff $p \in \mathbb{R}^N$, we define the *demand set* $D_i(p) \subseteq N$ by

$$D_i(p) = \arg\max\{v_{ij} - p_j \mid j \in N\}.$$

The conditions in (2.1) for an equilibrium can be rewritten using demand sets as follows.

Proposition 2.1. *For a matching $\mu : N \to N$ and a payoff $p \in \mathbb{R}^N$, the tuple (μ, p) is an equilibrium if and only if*

$$\mu(i) \in D_i(p), \quad p_i = \max\{v_{ij} - p_j \mid j \in N\} \geq 0 \qquad (\forall i \in N). \tag{2.2}$$

A necessary and sufficient condition for the existence of an equilibrium can be also given by using the following dual pair of linear programming problems:

$$
\begin{array}{lll}
\text{(P)} & \text{Maximize} & \displaystyle\sum_{i,j \in N, i<j} v_{ij} x_{ij} \\
& \text{subject to} & \displaystyle\sum_{j \in N, j>i} x_{ij} + \sum_{j \in N, j<i} x_{ji} \leq 1 \qquad (\forall i \in N), \\
& & x_{ij} \geq 0 \qquad (\forall i, j \in N, i<j), \\
\text{(D)} & \text{Minimize} & \displaystyle\sum_{i \in N} p_i \\
& \text{subject to} & p_i + p_j \geq v_{ij} \qquad (\forall i, j \in N, i<j), \\
& & p_i \geq 0 \qquad (\forall i \in N).
\end{array}
$$

(P) is a linear programming relaxation for the problem of finding a maximum-weight matching in (N, v), where the weight of a matching $\mu : N \to N$ is given by $\sum_{i \in N} v_{i,\mu(i)}$.

Proposition 2.2 (Talman and Yang (2011))**.** *For a partnership formation problem (N, v), the following conditions are equivalent.*
(a) *There exists an equilibrium in (N, v).*
(b) (P) *has an integral optimal solution.*
(c) *The maximum weight of a matching in (N, v) is equal to the optimal value of* (D)*.*

From Proposition 2.2, we can obtain the following properties.

Proposition 2.3 (cf. Talman and Yang (2011))**.** *Suppose that there exists an equilibrium in the partnership formation problem (N, v). Let $\mu : N \to N$ be a matching and $p \in \mathbb{R}^N$ a payoff.*
(i) *μ is an equilibrium matching if and only if it is a maximum-weight matching in (N, v).*
(ii) *p is an equilibrium payoff if and only if it is an optimal solution of the linear programming problem* (D)*.*
(iii) *If μ is an equilibrium matching and p is an equilibrium payoff, then (μ, p) is an equilibrium.*

The claim (iii) of Proposition 2.3 shows that a matching and a payoff in an equilibrium can be chosen independently of each other.

2.2. Assignment Game

An *assignment game* is represented by the tuple (A, B, w), where A, B are sets of agents corresponding to sellers and buyers, respectively, and $w =$

$(w(i,j) \mid i \in A,\ j \in B)$ ($\subseteq \mathbb{R}$) are (joint) values generated by pairs of agents $i \in A$ and $j \in B$. A *matching* in the assignment game (A, B, w) is a function $\eta : B \to A \cup \{0\}$ such that for each $i \in A$ there exists at most one $j \in B$ with $\eta(j) = i$. Here, 0 denotes a dummy seller that has no value (i.e., $w(0, j) = 0$ for $j \in B$) and can be a pair with any number of buyers, i.e., there may be distinct $j, j' \in B$ with $\eta(j) = \eta(j') = 0$. Vectors $q = (q_i \mid i \in A) \in \mathbb{R}^A$ and $r = (r_j \mid j \in B) \in \mathbb{R}^B$ are called *sellers' payoff* and *buyers' payoff*, respectively; the pair (q, r) is simply called a *payoff*.

For a matching $\eta : B \to A \cup \{0\}$, sellers' payoff $q \in \mathbb{R}^A$, and buyers' payoff $r \in \mathbb{R}^B$, the tuple (η, q, r) is called an *equilibrium* if the following conditions hold with $q_0 = 0$:

$$\left.\begin{array}{l} q_i + r_j \geq w(i,j) \qquad (\forall i \in A, j \in B), \\ q_{\eta(j)} + r_j = w(\eta(j), j) \qquad (\forall j \in B), \\ q_i \geq 0 \quad (\forall i \in A), \qquad r_j \geq 0 \quad (\forall j \in B), \\ q_i = 0 \qquad (\forall i \in A \setminus \{\eta(j) \mid j \in B\}). \end{array}\right\} \tag{2.3}$$

Every assignment game has an equilibrium (Shapley and Shubik, 1971), while in the partnership formation problem an equilibrium may not exist.

We see from the definition that an equilibrium of an assignment game (A, B, w) corresponds to an equilibrium of a partnership formation problem (N, v) such that $N = A \cup B$ and

$$v_{ij} = \begin{cases} w(i,j) & (\text{if } i \in A,\ j \in B \text{ or } j \in A,\ i \in B), \\ -\gamma & (\text{otherwise (i.e., } i, j \in A \text{ or } i, j \in B)), \end{cases}$$

where γ is an arbitrarily chosen positive number (see Talman and Yang (2011, Theorem 3); see also Chiappori et al. (2012)). That is, the assignment game can be reduced to the partnership formation problem.

A matching η and a payoff (q, r) in an equilibrium (η, q, r) are called an *equilibrium matching* and an *equilibrium payoff*, respectively. Equilibrium matching and payoff can be characterized as follows (see, e.g., Shapley and Shubik (1971); Roth and Sotomayor (1990)).

Proposition 2.4. *Let $\eta : B \to A \cup \{0\}$ be a matching of the assignment game (A, B, w), and $(q, r) \in \mathbb{R}^A \times \mathbb{R}^B$ be a payoff.*
(i) *η is an equilibrium matching if and only if it maximizes the weight $\sum_{j \in B} w(\eta(j), j)$ among all matchings.*
(ii) *(q, r) is an equilibrium payoff if and only if it is an optimal solution of*

the following linear programming problem:

$$
\begin{array}{ll}
\text{Minimize} & \displaystyle\sum_{i \in A} q_i + \sum_{j \in B} r_j \\
\text{subject to} & q_i + r_j \geq w(i,j) \quad (\forall i \in A, \forall j \in B), \\
& q_i \geq 0 \quad (\forall i \in A), \qquad r_j \geq 0 \quad (\forall j \in B).
\end{array}
$$

(iii) *If η is an equilibrium matching and (q, r) is an equilibrium payoff, then (η, q, r) is an equilibrium.*

The claim (iii) of the proposition above shows that a matching and a payoff in an equilibrium can be chosen independently of each other.

3. Reduction of Partnership Formation Problem to Assignment Game

As mentioned in Section 2, the assignment game can be reduced to the partnership formation problem. In this section, we show that the converse is also true in some sense, i.e., using an equilibrium of the assignment game, an equilibrium of the partnership formation problem can be obtained (if it exists). In Section 3.1 we explain the assignment game associated with a given partnership formation problem and show its fundamental properties. In Section 3.2 we show the main theorems on the relationship between the partnership formation problem and the associated assignment game. Base on the theorems, we present an algorithm for computing an equilibrium of the partnership formation problem.

3.1. Assignment Game Associated with Partnership Formation Problem

Given a partnership formation problem (N, v), we define an assignment game (N, N', w) as follows:

$$
\left.
\begin{array}{ll}
N' = \{i' \mid i \in N\}, \ \text{where } i' \text{ is a copy of } i, & \\
\text{for } i \in N, \ j' \in N' : \qquad w(i, j') = \begin{cases} v_{ij} & (\text{if } i \neq j), \\ 0 & (\text{if } i = j). \end{cases}
\end{array}
\right\} \tag{3.1}
$$

The assignment game defined above is essentially the same as the one in Chiappori et al. (2012). In the following discussion, we often identify the set N' with N through a natural one-to-one correspondence, and regard a vector in $\mathbb{R}^{N'}$ (resp., $\mathbb{R}^{N}$) as a vector in $\mathbb{R}^{N}$ (resp., $\mathbb{R}^{N'}$).

Due to the symmetric structure of the assignment game (N, N', w), its equilibria have various nice properties. The next property shows that (N, N', w) has an equilibrium matching in which no buyer in N' is assigned to dummy item and every seller in N is assigned to some buyer in N'.

Proposition 3.1. *There exists an equilibrium matching $\eta : N' \to N \cup \{0\}$ in (N, N', w) such that η is a bijection from N' to N.*

Proof. We prove the claim by a graph-theoretic argument. Let us consider a complete bipartite graph $G = (N, N'; N \times N')$ on the vertex set $N \cup N'$ with edge weight given by $w(i, j')$ ($(i, j') \in N \times N'$). Recall that a matching in G is a set M of edges such that for each vertex i there exists at most one edge in M incident to i. It is easy to see that matchings in (N, N', w) have a natural one-to-one correspondence with matchings in G; moreover, by Proposition 2.4 (i), a matching in (N, N', w) is an equilibrium matching if and only if its corresponding matching in G is a maximum-weight matching in G. Hence, it suffices to show that there exists a maximum-weight matching in G that is a perfect matching (i.e., a matching that covers all vertices in G).

Let $M \subseteq N \times N'$ be a maximum-weight matching in G. We may assume that M is not a perfect matching. Since $w(i, j') = w(j, i')$ holds for $i, j \in N$, the matching M' given by

$$M' = \{(j, i') \in N \times N' \mid (i, j') \in M\}$$

is also a maximum-weight matching in G.

Using the two matchings M and M', we define an edge set X as follows. For a vertex i in the graph G, we denote by $\deg_X(i)$ the number of edges in X incident to i. We initially set $X = M \cup M'$. Then, we have $\deg_X(i) = \deg_X(i') \leq 2$ for $i \in N$ and $i' \in N'$. For each $i \in N$, if $\deg_X(i) = \deg_X(i') = 1$ then we add to X (one copy of) the edge (i, i'), and if $\deg_X(i) = \deg_X(i') = 0$ then we add to X two copies of the edge (i, i'). Then, the resulting edge set X satisfies $\deg_X(i) = \deg_X(i') = 2$ for each $i \in N$. This implies that X can be decomposed into two perfect matchings, which are denoted as X_1 and X_2. Moreover, the total weight of the edge set X is twice as much as the weight of a maximum-weight matching since X contains all edges in $M \cup M'$ and each edge (i, i') has zero weight. Hence, both of the matchings X_1 and X_2 are maximum-weight matchings that are perfect matchings. □

Based on this observation, in the following discussion we restrict matchings in (N, N', w) to bijections from N' to N.

We present some properties of equilibria. For a vector $q \in \mathbb{R}^N$ and $j' \in N'$, we define a set $\widetilde{D}_{j'}(q) \subseteq N$ by

$$\widetilde{D}_{j'}(q) = \arg\max\{w(i, j') - q_i \mid i \in N\}. \tag{3.2}$$

By definition, we have $\widetilde{D}_{j'}(q) = D_j(q)$. Let

$$\mathcal{H}_0 = \{q \in \mathbb{R}^N \mid (q, r) \text{ is an equilibrium payoff in } (N, N', w) \text{ for some } r \in \mathbb{R}^{N'}\}. \tag{3.3}$$

Proposition 3.2. *Let $\eta : N' \to N$ be a (bijection) matching in (N, N', w).* (i) *A payoff $(q, r) \in \mathbb{R}^N \times \mathbb{R}^{N'}$ is an equilibrium payoff in (N, N', w) if and only if $q \in \mathcal{H}_0$ and*

$$r_{j'} = \max\{w(i, j') - q_i \mid i \in N\} \quad (j' \in N'). \tag{3.4}$$

(ii) *Suppose that $q \in \mathcal{H}_0$. Then, η is an equilibrium matching in (N, N', w) if and only if $\eta(j') \in \widetilde{D}_{j'}(q)$ $(\forall j' \in N)$.*

Proof. By definition, the tuple (η, q, r) is an equilibrium in (N, N', w) if and only if

$$q_i + r_{j'} \geq w(i, j') \qquad (\forall i \in N, j' \in N'), \tag{3.5}$$

$$q_{\eta(j')} + r_{j'} = w(\eta(j'), j') \qquad (\forall j' \in N'), \tag{3.6}$$

$$q_i \geq 0 \quad (\forall i \in N), \qquad r_{j'} \geq 0 \quad (\forall j' \in N'), \tag{3.7}$$

Hence, the claim (i) follows from (3.5) and (3.6). Also, the claim (ii) follows from (3.5), (3.6), and Propositions 2.4 (iii). □

The next property shows that there exists an equilibrium payoff of the form (p, p) in (N, N', w).

Proposition 3.3. *For an equilibrium payoff (q, r) in (N, N', w), $((q + r)/2, (q + r)/2)$ is also an equilibrium payoff in (N, N', w).*

Proof. By Proposition 2.4 (ii) and the symmetry of the assignment game (N, N', w), the payoff (r, q) is an equilibrium payoff in (N, N', w). Since the set of equilibrium payoffs is a convex set by Proposition 2.4 (ii), $((q + r)/2, (q + r)/2)$ is also an equilibrium payoff. □

3.2. Theorem and Algorithm

We first show that an equilibrium in a partnership formation problem (N, v) corresponds to a "symmetric" equilibrium in the associated assignment game (N, N', w). For a matching $\eta : N' \to N$ in (N, N', w), we denote by $\eta^{-1} : N' \to N$ the matching in (N, N', w) such that $\eta^{-1}(j') = i$ if $\eta(i') = j$. We say that an equilibrium (η, q, r) in (N, N', w) is *symmetric* if $\eta = \eta^{-1}$ and $q = r$ hold.

For a matching $\mu : N \to N$ in (N, v), the matching $\eta_\mu : N' \to N$ in (N, N', w) associated with μ is given by

$$\eta_\mu(j') = \mu(j) \qquad (j' \in N'). \tag{3.8}$$

Note that the matching η_μ satisfies the condition $\eta_\mu = (\eta_\mu)^{-1}$.

Theorem 3.4. *For a matching $\mu : N \to N$ and a payoff $p \in \mathbb{R}^N$ in (N, v), the tuple (μ, p) is an equilibrium in (N, v) if and only if (η_μ, p, p) with the matching $\eta_\mu : N' \to N$ given by (3.8) is an equilibrium in (N, N', w). In particular, there exists an equilibrium in (N, v) if and only if there exists a symmetric equilibrium in (N, N', w).*

Proof. The conditions (2.1) for (μ, p) to be an equilibrium in (N, v) can be rewritten in terms of η_μ, p, and $w(i, j')$ as follows:

$$
\begin{aligned}
& p_i + p_{j'} \geq w(i, j') \qquad (\forall i \in N, \forall j' \in N'), \\
& p_{\eta_\mu(j')} + p_{j'} = w(\eta_\mu(j'), j') \qquad (\forall j' \in N'), \\
& p_i \geq 0 \qquad (\forall i \in N).
\end{aligned}
$$

These conditions hold if and only if (η_μ, p, p) is an equilibrium in (N, N', w). □

We then show that an equilibrium of the partnership formation problem (N, v) can be obtained by using an equilibrium payoff of the associated assignment game (N, N', w), provided that an equilibrium exists in (N, v).

Theorem 3.5. *Let $\mu : N \to N$ be a matching in (N, v) and $q \in \mathcal{H}_0$.*
(i) *μ is an equilibrium matching in (N, v) if and only if*

$$\mu(j) \in D_j(q) \qquad (\forall j \in N). \tag{3.9}$$

(ii) *Suppose that μ is an equilibrium matching in (N, v). Define $p \in \mathbb{R}^N$ by*

$$p_j = (1/2)(v_{j,\mu(j)} + q_j - q_{\mu(j)}) \qquad (j \in N). \tag{3.10}$$

Then, p is an equilibrium payoff in (N, v).

Proof. Let $\eta_\mu : N' \to N$ be the matching in (N, N', w) given by (3.8). Below we omit the subscript of η_μ for simplicity.

We first prove the claim (i). By Theorem 3.4, μ is an equilibrium matching in (N, v) if and only if η is an equilibrium matching in (N, N', w). By Proposition 3.2 (ii), η is an equilibrium matching in (N, N', w) if and only if $\eta(j') \in \widetilde{D}_{j'}(q)$ $(\forall j' \in N')$, which is equivalent to the condition (3.9) since $D_j(q) = \widetilde{D}_{j'}(q)$ and $\mu(j) = \eta(j')$ hold for $j \in N$.

We then prove the claim (ii). By Theorem 3.4, η is an equilibrium matching in (N, N', w). It follows from Proposition 3.2 (i) and 3.3, that (η, q, r) is an equilibrium in (N, N', w) with $r \in \mathbb{R}^{N'}$ given as $r_{j'} = w(\eta(j'), j') - q_{\eta(j')}$

($j' \in N'$). By Proposition 2.4 (iii), $(\eta, (q+r)/2, (q+r)/2)$ is also an equilibrium in (N, N', w), from which follows that $(\mu, (q+r)/2)$ is an equilibrium in (N, v) by Theorem 3.4. By the definitions of η and w, we have

$$\begin{aligned}(1/2)(q_j + r_{j'}) &= (1/2)(q_j + w(\eta(j'), j') - q_{\eta(j')}) \\ &= (1/2)(q_j + v_{j,\mu(j)} - q_{\mu(j)}) = p_j \qquad (j \in N),\end{aligned}$$

implying that p is an equilibrium payoff in (N, v). □

Based on Theorem 3.5, we can check the existence of an equilibrium in a partnership formation problem (N, v) by the following algorithm. The idea is as follows. Let $q \in \mathbb{R}^N$ be buyers' payoff in $\mathcal{H}_0$. If (N, v) has an equilibrium, then Theorem 3.5 implies that there exists a matching $\mu : N \to N$ in (N, v) satisfying the condition (3.9), which is an equilibrium matching. Therefore, if there exists no matching satisfying (3.9), then we can discern that (N, v) has no equilibrium.

Algorithm ComputeEquilibrium
Step 1: Find a vector $q \in \mathcal{H}_0$.
Step 2: If there exists no matching $\mu : N \to N$ in (N, v) satisfying (3.9), then assert that "no equilibrium exists in (N, v)" and stop.
Step 3: Find a matching $\mu : N \to N$ in (N, v) satisfying (3.9), and let $p \in \mathbb{R}^N$ be a vector given by (3.10).
Output (μ, p) as an equilibrium of (N, v). □

Remark 3.6. We discuss the computation of a vector $q \in \mathcal{H}_0$ in Step 1. Suppose that we can obtain the information on the sets

$$\arg\max\{w(i, j') - q_i \mid i \in N \cup \{0\}\} \qquad (j' \in N', q \in \mathbb{R}^N).$$

Note that this set contains the set $\widetilde{D}_{j'}(q)$ if $\max\{w(i, j') - q_i \mid i \in N\} \geq 0$ and this set is equal to $\{0\}$ if $\max\{w(i, j') - q_i \mid i \in N\} < 0$. Then, a vector $q \in \mathcal{H}_0$ can be computed by any of price adjustment processes for the assignment game (see, e.g., Andersson et al. (2010); Andersson and Erlanson (2013); Demange et al. (1986); Mishra and Parkes (2009)). In particular, minimal and maximal vectors in $\mathcal{H}_0$ can be computed.

Even in the case where only the information of the sets $\widetilde{D}_{j'}(q) = D_j(q)$ is available, a vector $q \in \mathcal{H}_0$ can be computed by a price adjustment process by Andersson et al. (2014a). See Section 4 for details. □

Remark 3.7. Computation of a matching μ in (N, v) satisfying (3.9) can be reduced to the problem of finding a maximum-cardinality matching in a undirected graph with vertex set N and edge set given by $\{(i,j) \mid i,j \in N,\ i \in D_j(p)\}$ (see Andersson et al. (2014a)). Note that finding a maximum-cardinality matching in an undirected graph can be done in strongly polynomial time (see, e.g., Schrijver (2003)). □

4. Connection to Algorithm of Andersson et al. (2014a)

We show that our algorithm in Section 3 can be seen as a generalization of the algorithm in Andersson et al. (2014a). In particular, the choice of a vector $q \in \mathcal{H}_0$ in Step 1 is more flexible in our algorithm, while it is fixed to a (unique) minimal vector in $\mathcal{H}_0$ in Andersson et al. (2014a), as proved below.

In this section, we assume that the values v_{ij} are integers. For a payoff $p \in \mathbb{R}^N$ and a set $S \subseteq N$ of agents, we define

$$O(S,p) = \{j \in N \mid D_j(p) \subseteq S\},$$
$$U(S,p) = \{j \in N \mid D_j(p) \cap S \neq \emptyset\}.$$

We say that S is *in excess demand* if the following condition holds:

$$|U(T,p) \cap O(S,p)| > |T| \qquad (\emptyset \neq \forall T \subsetneq S).$$

Note that if $|O(S,p)| > |S|$ holds for some $S \subseteq N$, then a set in excess demand exists, and a maximal set in excess demand is uniquely determined (see Andersson et al. (2010, 2013); Mo et al. (1988)).

We describe the algorithm of Andersson et al. (2014a); the following is a variant of the original algorithm given in Andersson et al. (2014b).

Algorithm PartnershipFormationProcess
Step 0: Set $p := (0, 0, \ldots, 0)$.
Step 1: Collect the demand sets $D_j(p)$ for $j \in N$.
Step 2: If $|O(S,p)| \leq |S|$ ($\forall S \subseteq N$), then set $\widehat{p} = p$ and go to Step 4.
Step 3: Find the unique maximal set $S \subseteq A$ in excess demand, update p by $p_i := p_i + 1$ ($i \in S$), and go to Step 1.
Step 4: If there exists a matching $\mu : N \to N$ such that $\mu(j) \in D_j(\widehat{p})$ ($\forall j \in N$), then collect the values $v_{j,\mu(j)}$ for $j \in N$, compute the payoff $p^* \in \mathbb{R}^N$ by

$$p^*_j = (1/2)(v_{j,\mu(j)} + \widehat{p}_j - \widehat{p}_{\mu(j)}) \qquad (j \in N),$$

and output (μ, p^*) as an equilibrium of (N, v).
Otherwise, assert that "no equilibrium exists." □

We will show that the vector $\widehat{p}$ is contained in $\mathcal{H}_0$. Hence, Algorithm PartnershipFormationProcess can be seen as an implementation of our algorithm with a special selection of a vector $q \in \mathcal{H}_0$ in Step 1. Moreover, we prove the following stronger statement:

Theorem 4.1. *The vector $\widehat{p}$ found in Algorithm* PartnershipFormationProcess *is equal to a (unique) minimal vector in $\mathcal{H}_0$.*

Note that a minimal vector in the set $\mathcal{H}_0$ is uniquely determined (see, e.g., Shapley and Shubik (1971)).

To prove Theorem 4.1, the following property is crucial. We define

$$\mathcal{H} = \{p \in \mathbb{R}^N \mid p_i \geq 0 \ (\forall i \in N), \ |O(S,p)| \leq |S| \ (\forall S \subseteq N)\}. \tag{4.1}$$

Proposition 4.2 (Andersson et al. (2014a, Theorem 2))**.** *A minimal vector in $\mathcal{H}$ is uniquely determined and equal to the vector $\widehat{p}$.*

We then rewrite the definition of $\mathcal{H}$ in terms of the assignment game (N, N', w). Recall the definition of $\widetilde{D}_{j'}(q)$ in (3.2).

Proposition 4.3.

$$\mathcal{H} = \{q \in \mathbb{R}^N \mid q_i \geq 0 \ (\forall i \in N), \ \exists \textit{matching } \eta : N' \to N \textit{ in } (N, N', w) \\ \textit{s.t. } \eta(j') \in \widetilde{D}_{j'}(q) \ (\forall j' \in N')\}. \tag{4.2}$$

Proof. For every $S \subseteq N$ and $q \in \mathbb{R}^N$, we have $U(N \setminus S, q) = N \setminus O(S, q)$. Hence, $|O(S,q)| \leq |S|$ holds if and only if

$$|U(N \setminus S, q)| = |N| - |O(S,q)| \geq |N| - |S| = |N \setminus S|,$$

implying that

$$\mathcal{H} = \{q \in \mathbb{R}^N \mid q \geq 0, \ |U(T,p)| \geq |T| \ (\forall T \subseteq N)\}. \tag{4.3}$$

For $T \subseteq N$ and $q \in \mathbb{R}^N$, we define

$$\widetilde{U}(T,q) = \{j' \in N' \mid \widetilde{D}_{j'}(q) \cap T \neq \emptyset\} \ (= \{j' \in N' \mid j \in U(T,q)\}).$$

Since $|\widetilde{U}(T,q)| = |U(T,q)|$, we can rewrite the formula (4.3) as

$$\mathcal{H} = \{q \in \mathbb{R}^N \mid q \geq 0, \ |\widetilde{U}(T,p)| \geq |T| \ (\forall T \subseteq N)\}.$$

By the well-known Hall's theorem in graph theory, the condition $|\widetilde{U}(T,q)| \geq |T|$ $(\forall T \subseteq N)$ holds if and only if there exists some matching $\eta : N' \to N$ such that $\eta(j') \in \widetilde{D}_{j'}(q)$ $(\forall j' \in N')$. Hence, we obtain (4.2). □

This property implies the following relation between $\mathcal{H}$ and $\mathcal{H}_0$.

Proposition 4.4.

$$\mathcal{H}_0 = \{q \in \mathcal{H} \mid \max\{w(i,j') - q_i \mid i \in N\} \geq 0 \ (\forall j' \in N')\}. \tag{4.4}$$

Proof. By the conditions for equilibrium in (N, N', w) (cf. (3.5), (3.6), (3.7)) we have $q \in \mathcal{H}_0$ if and only if the following conditions hold:

$$\begin{aligned}
&\max\{w(i,j') - q_i \mid i \in N\} \geq 0 \qquad (\forall j' \in N'),\\
&\exists\,\text{matching } \eta : N' \to N \text{ s.t. } \eta(j') \in \widetilde{D}_{j'}(q) \ (\forall j' \in N),\\
&q_i \geq 0 \qquad (\forall i \in N).
\end{aligned}$$

This, together with Proposition 4.3, implies (4.4). □

We now prove Theorem 4.1. By Proposition 4.4, it holds that $\widehat{q} \in \mathcal{H}_0 \subseteq \mathcal{H}$. Since $\widehat{p}$ is the unique minimal vector in $\mathcal{H}$, we have $\widehat{p} \leq \widehat{q}$, which implies

$$\max\{w(i,j') - \widehat{p}_i \mid i \in N\} \geq \max\{w(i,j') - \widehat{q}_i \mid i \in N\} \geq 0,$$

where the last inequality is by Proposition 4.4. Hence, we have $\widehat{p} \in \mathcal{H}_0$ by Proposition 4.4. By the minimality of $\widehat{q}$ in $\mathcal{H}_0$ and $\widehat{p} \geq \widehat{q}$, we have $\widehat{p} = \widehat{q}$. This concludes the proof of Theorem 4.1.

Remark 4.5. By Theorem 4.1, Steps 1–3 of Algorithm PartnershipFormationProcess can be seen as a price adjustment process to compute the unique minimal vector in $\mathcal{H}_0$. Moreover, the behavior of the vector q during the iterations of Steps 1–3 is the same as that of the price adjustment process by Demange et al. (1986) and Mo et al. (1988) applied to the assignment game (N, N', w). □

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