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A Simple and Procedurally Fair Game Form for Nash Implementation of the No-envy Solution*

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Abstract

We consider the allocation problem of infinitely divisible resources with at least three agents. For this problem, Thomson (*Games and Economic Behavior* 52: 186-200, 2005) and Doğan (*Games and Economic Behavior*, 98: 165-171, 2016) propose "simple" but not "procedurally fair" game forms which implement the "no-envy" solution in Nash equilibria. By contrast, Galbiati (*Economics Letters*, 100: 72-75, 2008) constructs a procedurally fair but not simple game form which implements the no-envy solution in Nash equilibria. In this paper, we design a both simple and procedurally fair game form which implements the no-envy solution in Nash equilibria.

JEL Classification: C72, D71, D78

Keywords: Simple game form; Procedural fairness; Nash implementation; No-envy solution

1 Introduction

We consider the allocation problem of infinitely divisible resources with at least three agents. For this problem, Thomson [19] and Doğan [5] propose "simple" but not "procedurally fair" game forms which implement the "no-envy" solution in Nash equilibria.

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By contrast, Galbiati [7] constructs a procedurally fair but not simple game form which implements the no-envy solution in Nash equilibria. In this paper, we investigate the possibility of *Nash implementation* of the no-envy solution by a game form that is both simple and procedurally fair. Note that in Doğan [5] and this study, there is no monotonicity assumption on preferences, although Thomson [19] and Galbiati [7] make the assumption of "strict monotonicity"on preferences.[1] For detailed discussions concerning simplicity and procedural fairness of game forms, see Section 3.

In studying the implementation problem, the objective of a social planner is embodied by a "solution". Mathematically, a solution is a set-valued mapping which, for each possible preference profile, specifies a non-empty set of outcomes.[2] Each agent knows the other agents' preferences, but the planner does not. Then, the planner specifies a message space for each agent and a mapping which, for each possible message profile, chooses an outcome. The pair consisting of the list of message spaces and the mapping is a "game form".

In the allocation problem of infinitely divisible resources, the planner selects an allocation in which the summation of assignments for all agents is equal to a social endowment, i.e., the balance for the social endowment is satisfied, and she assigns a bundle of the allocation to each agent. We want this allocation to be "envy-free": no agent prefers the bundle of a different agent over his own bundle (Foley [6]). The "no-envy"solution selects the *envy-free* allocations for each preference profile.

We construct a game form for *Nash implementation* of the no-envy solution. We call it "Choose-Two-Bundles-and-Transpose". In Choose-Two-Bundles-and-Transpose, each

[1] In the same model as Galbiati [7], Saijo et al. [16] also constructs procedurally fair game forms to implement the no-envy solution in Nash equilibria. However, the game form of Galbiati [7] is simpler than game forms of Saijo et al. [16].

[2] A solution is also called a "social choice correspondence".

agent announces two bundles each of which is possible consumption bundle for the social endowment as well as the name of an agent. The two bundles are interpreted as the first bundle is for his own assignment, and the second bundle is for his neighbor's assignment.[3] Reporting a name by an agent can be interpreted as this agent wants to exchange with the bundle of the named agent. Therefore, the message can be natural. The outcome mapping is as follows: If the second bundle reported by each agent is the same as the first bundle reported by his neighbor and the list of bundles based on the announcements of agents is balanced for the social endowment, then each agent gets one bundle of the transposed allocation. If there is only one agent that reports a different bundle from the message reported by his neighbor and the list of bundles based on announcements of the other agents is balanced for the social endowment, then each agent gets one bundle of the transposed allocation. Therefore, a message concerning bundles reported by a deviator will be ignored. Otherwise, each agent gets one bundle of the equal-division allocation. Note that this game form depends on the existence of the equal-division allocation.[4] By contrast, for the game forms of Thomson [19] and Galbiati [7], it is important that there exists a least preferred bundle by the assumption of *strict monotonicity* on preferences.

We show that in the allocation problem of infinitely divisible resources, if there are at least three agents, Choose-Two-Bundles-and-Transpose implements the no-envy solution in Nash equilibria (Theorem 1). Our result is applicable in, for example, the cake division

[3]This interpretation is related to Saijo [15] and Saijo et al. [16]. In the game form of Saijo [15], each agent reports his own preference and his neighbor's. Although the idea of ordering the agents in a circular fashion and letting each of them report a message for the next agent in the circle is the same as in our game form, the message spaces applied to this idea in the game form of Saijo [15] are different from those in our game form, and our game form is simpler than that of Saijo [15]. In the game form of Saijo et al. [16], each agent reports only two bundles each of which is possible consumption bundle for the social endowment. Although they apply the above same idea as in Saijo [15] and this study, our game form is simpler than that of Saijo et al. [16].

[4]Since Choose-Two-Bundles-and-Transpose depends on the existence of the equal-division allocation, this game form is not applicable to a model in which there is an indivisible good.

problem (Thomson [20]) and the allocation problem of infinitely divisible resources with single-peaked preferences (Adachi [1], Morimoto et al. [14]).[5]

This paper is organized as follows. Section 2 provides a game form for *Nash implementation* of the no-envy solution in the allocation problem of infinitely divisible resources. Section 3 reports related literature. Section 4 proposes concluding remarks.

2 Allocation Problems of Infinitely Divisible Resources

Let $N = \{1, ..., n\}$ be a **set of agents** among whom a **social endowment** $\Omega \in \mathbb{R}^{\ell}_{++}$ of ℓ infinitely divisible resources has to be allocated. We assume that the resources cannot be disposed of. An **allocation for** $\Omega \in \mathbb{R}^{\ell}_{++}$ is a list $a = (a_1, \cdots, a_n) \in \mathbb{R}^{\ell n}_{+}$ such that $\Sigma_{i \in N} a_i = \Omega$. Let $A^{\Omega} = \{a \in \mathbb{R}^{\ell n}_{+} : \Sigma_{i \in N} a_i = \Omega\}$ be the **set of allocations for** $\Omega \in \mathbb{R}^{\ell}_{++}$. Let $X^{\Omega} = \{x \in \mathbb{R}^{\ell}_{+} : x \leq \Omega\}$ be the **set of possible consumption bundles for** $\Omega \in \mathbb{R}^{\ell}_{++}$.[6] Let $\tilde{a} \equiv \frac{\Omega}{|N|} \in A^{\Omega}$ be the **equal-division allocation for** $\Omega \in \mathbb{R}^{\ell}_{++}$. Let $R_i \in \mathcal{R}_i$ be a **preference for agent** $i \in N$ **over** X^{Ω}, where $\mathcal{R}_i$ is the **set of preferences admissible for agent** i. Let $R = (R_1, ..., R_n) \in \mathcal{R}$ be a **preference profile**, where $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$. Note that for each $i \in N$, each $R_i \in \mathcal{R}_i$, and each pair $a, b \in A^{\Omega}$ such that $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$, $a_i \; R_i \; b_i$ iff $a \; R_i \; b$. Then, each agent's preferences over X^{Ω} are extended to over A^{Ω}. Let (N, Ω, R) be the **allocation problem of infinitely divisible resources**. We fix N and Ω, so that the problem is represented by R.

[5] In the allocation problem of an infinitely divisible resource with single-peaked preferences (Sprumont [17]), Thomson [21] addresses *Nash implementability* of several solutions which do not satisfy "no veto power" in this model, in particular the no-envy solution. Since *no veto power* is one of sufficient conditions for *Nash implementability* of solutions (Saijo [16]), the game form of Saijo [16] is not applicable in this model. Then, Thomson [21] shows that the no-envy solution is *Nash implementable* by the result of Yamato [22]. However, he does not propose any simple game form.

[6] Given $x, y \in \mathbb{R}^{\ell}_{+}$, $x \leq y$ means that for each $j \in \{1, \cdots, \ell\}$, $x_j \leq y_j$.

An allocation $a \in A^{\Omega}$ is **envy-free for** $R \in \mathcal{R}$ if for each $\{i,j\} \subseteq N$, $a_i \; R_i \; a_j$. The **no-envy solution** $F : \mathcal{R} \twoheadrightarrow A^{\Omega}$ is a correspondence which, for each $R \in \mathcal{R}$, $F(R)$ is the set of *envy-free* allocations for R.

In order to design our game form to implement the no-envy solution in Nash equilibria, let us introduce a definition. Let $T^i_j : A^{\Omega} \to A^{\Omega}$ be a **transposition mapping** which, for each $a \in A^{\Omega}$, selects the allocation by transposing the bundles of agent i and agent j in a.

"Choose-Two-Bundles-and-Transpose" is a game form constructed for *Nash implementation* of the no-envy solution. In Choose-Two-Bundles-and-Transpose, each agent $i \in N$ announces two bundles, $x^i, y^i \in X^{\Omega}$, and the name of an agent, $k^i \in N$. The outcome mapping is as follows: If the second bundle y_i reported by an agent, $i \in N$, is the same as the first bundle x^{i+1} reported by his neighbor, $i+1$, i.e., $y^i = x^{i+1} \equiv a_{i+1}$, and the list of bundles based on announcements of agents is balanced for the social endowment, i.e., $(a_1, \cdots, a_n) \in A^{\Omega}$, then each agent gets one bundle of the transposed allocation, i.e., $T^{k^n}_n \circ \cdots T^{k^1}_1 \circ (a_1, \cdots, a_n)$. If there is only one agent $i \in N$ reports a different bundle from the message reported by his neighbor, $i-1$ or $i+1$, and the list of bundles based on announcements of the other agents is balanced for the social endowment, i.e., $(a_1, \cdots, y^{i-1}, x^{i+1}, \cdots, a_n) \in A^{\Omega}$, then each agent gets one bundle of the transposed allocation, i.e., $T^{k^n}_n \circ \cdots T^{k^1}_1 \circ (a_1, \cdots, y^{i-1}, x^{i+1}, \cdots, a_n)$. Therefore, a message concerning bundles reported by a deviator will be ignored. Otherwise, each agent gets one bundle of the equal-division allocation.

Choose-Two-Bundles-and-Transpose, $\Gamma^{C2T} = (M, g)$**:** For each $i \in N$, $M_i = X^{\Omega} \times X^{\Omega} \times N$. Given $m = (x^i, y^i, k^i)_{i \in N} \in \times_{i \in N} M_i \equiv M$, the outcome mapping

$g : M \to A^{\Omega}$ is as follows:[7]

$$g(m) = \begin{cases} \text{Rule 1: } T_n^{k^n} \circ \cdots T_1^{k^1} \circ (a_1, \cdots, a_n) & \text{if } \begin{cases} \text{for each } i \in N, y^i = x^{i+1} \equiv a_{i+1}, \text{ and} \\ (a_1, \cdots, a_n) \in A^{\Omega} \end{cases} \\ \text{Rule 2: } T_n^{k^n} \circ \cdots T_1^{k^1} \circ (a_1, \cdots, y^{i-1}, x^{i+1}, \cdots, a_n) & \text{if } \begin{cases} \text{there is } i \in N \text{ such that for each } j \neq i, \\ y^j = x^{j+1} \equiv a_{j+1} \text{ and } y^{i-1} \neq x^i \\ \text{ or } y^i \neq x^{i+1}, \text{ and} \\ (a_1, \cdots, y^{i-1}, x^{i+1}, \cdots, a_n) \in A^{\Omega} \end{cases} \\ \text{Rule 3: } \tilde{a} & \text{otherwise} \end{cases}$$

Let (Γ^{C2T}, R) be the **game** induced by Γ^{C2T} and $R \in \mathcal{R}$. A message profile $m \in M$ is a **Nash equilibrium of** (Γ^{C2T}, R) if for each $i \in N$ and each $m_i' \in M_i$, $g(m_i, m_{-i})\ R_i\ g(m_i', m_{-i})$. Let $NE(\Gamma^{C2T}, R)$ be the **set of Nash equilibria of** (Γ^{C2T}, R).

The game form Γ^{C2T} **implements the no-envy solution** F **in Nash equilibria** if for each $R \in \mathcal{R}$, $F(R) = g(NE(\Gamma^{C2T}, R))$.

The following is our main result.

Theorem 1. *Let $n \geq 3$. Choose-Two-Bundles-and-Transpose Γ^{C2T} implements the no-envy solution F in Nash equilibria.*

Proof. Let $R \in \mathcal{R}$. We prove it by two steps.

Step 1. $F(R) \subseteq g(NE(\Gamma^{C2T}, R))$.

Let $a = (a_1, \cdots, a_n) \in F(R)$ and $m = (a_i, a_{i+1}, i)_{i \in N}$. By Rule 1, $g(m) = a$. For each $i \in N$, let $m_i' \neq m_i$. By Rule 1 or 2, $g_i(m_i', m_{-i}) \in \{a_1, \cdots, a_n\}$. Since $a \in F(R)$, $g_i(m_i, m_{-i}) = a_i\ R_i\ g_i(m_i', m_{-i})$. Therefore, for each $i \in N$ and each $m_i' \in M_i$, $g_i(m_i, m_{-i})\ R_i\ g_i(m_i', m_{-i})$. Hence, $m \in NE(\Gamma^{C2T}, R)$.

[7]Suppose that $a_{1-1} = a_n$ and $a_{n+1} = a_1$.

Step 2. $g(NE(\Gamma^{C2T}, R)) \subseteq F(R)$.

We show that if $g(m) \notin F(R)$, then $m \notin NE(\Gamma^{C2T}, R)$. Let $g(m) = a$ and $m = (x^i, y^i, k^i)_{i \in N}$. Since $a \notin F(R)$ and $\tilde{a} \in F(R)$, Rule 1 or 2 is applied. Since $a \notin F(R)$, there is a pair $\{i, j\} \subset N$ such that $a_j \ P_i \ a_i$. By selecting $m_i' = (x^i, y^i, k'^i)$ appropriately, $g_i(m_i', m_{-i}) = a_j$. Hence, $g_i(m_i', m_{-i}) = a_j \ P_i \ a_i = g_i(m_i, m_{-i})$. Therefore, $m \notin NE(\Gamma^{C2T}, R)$.■

3 Related Literature

We first consider the allocation problem of infinitely divisible resources with "strictly monotonic" preferences. For each $i \in N$, a preference R_i is **strictly monotonic** if for each pair $x, y \in X^{\Omega}$, $x \geq y$ and $x \neq y$ imply $x \ P_i \ y$. For the problem with only two agents, a well-known game form for *Nash implementation* of the no-envy solution is "Divide-and-Choose". One agent divides the resource into two parts, and the other agent chooses one of them. Although Divide-and-Choose is simple, this game form works well only in the case of two agents.

For the allocation problem with strictly monotonic preferences and at least two agents, "Divide-and-Permute" implements the no-envy solution in Nash equilibria (Thomson [19]). This game form resembles Divide-and-Choose. Although Divide-and-Permute is simple and works well with at least two agents, this game form is only applicable to models where the first and second agents always prefer any bundle to the bundle receiving nothing.

For the allocation problem in which such least-preferred bundles do not necessary exist, if there are at least three agents, "Divide-and-Transpose" implements the no-envy solution in Nash equilibria (Doğan [5]).[8,9] This game form is a modification of Divide-and-Permute. Although Divide-and-Transpose is simple and applicable to models without a monotonic condition on preferences, this game form does not treat all agents equally. More formally, this game form is not "ex ante fair"(Korpela [12])[10]: a game form $\Gamma = (S, h)$, where $h : S \to A^{\Omega}$, is *ex ante fair* if for each message profile $s \in S$ and each one-to-one function $\pi : N \to N$, there is another message profile $s' \in S$ such that $h(s') = \pi(h(s))$ and for each $i \in N$, $h(S_i, s'_{-i}) = \pi(h(S_{\pi(i)}, s_{-\pi(i)}))$.[11]

For the allocation problems with strictly monotonic preferences and at least two agents, "Galbiati's game form"implements the no-envy solution in Nash equilibria (Galbiati [7]). This game form is another modification of Divide-and-Permutate. In Galbiati's game form, each agent proposes an allocation, a one-to-one function from N to N, and the names of two agents. Although Galbiati's game form treats all agents equally so that it is *ex ante fair*, the message space is large. For example, suppose that there are ten agents and three types of resources. Each agent reports at least twenty-seven real-numbers for the other agents' assignments in addition to three real-numbers for his own assignments.

We designed a simple and *ex ante fair* game form, Choose-Two-Bundles-and-Transpose, to implement the no-envy solution in Nash equilibria.

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[8]In Divide-and-Transpose, each of the first, second, and third agents reports an allocation and the names of two agents, and each of the other agents only reports the names of two agents. However, it is enough for each agent to report the name of an agent.

[9]Even if there is an indivisible good, Divide-and-Transpose works well. For example, the result concerning *Nash implementation* by this game form can be applied to the allocation problem of indivisible objects with monetary transfers (e.g., Svensson [18]).

[10]As Korpela [12] states that Divide-and-Permute is not *ex ante fair*, we also easily check for Divide-and-Transpose not being *ex ante fair*.

[11]For each $a \in A^{\Omega}$, let $\pi(a) = (a_{\pi(1)}, \cdots, a_{\pi(n)})$. Given $s'_{-i} \in S_{-i}$, let $h(S_i, s'_{-i}) = \{h(s_i, s'_{-i}) : s_i \in S_i\}$. For each $A' \subseteq A^{\Omega}$, let $\pi(A') = \{\pi(a) : a \in A'\}$.

4 Concluding Remarks

For implementation theory, simple game forms are important. If a game form is complicated, and an agent does not understand how to select outcomes, then even if he wants to achieve the best outcome, he may not choose a message that induces the best outcome for his preference over the set of attainable.

"Strategy-proofness" requires that in the direct game form associated with the single-valued solution, for each agent, truth-telling is a dominant strategy. Since the objective of the planner is achieved at a dominant strategy equilibrium, *strategy-proofness* is desirable. However, laboratory experiments concerning *strategy-proof* single-valued solutions reported that in some games, some subjects did not select dominant strategies.[12] For example, in second-price-auction experiments, most bidders did not reveal true values (Harstad [9], Kagel and Levin [11], and Kagel et al.[10]). In an ascending auction and a second-price auction, subjects were substantially more likely to play truth-telling under the former than under the latter (Kagel et al. [10]). Inspired from these observations, "obvious" *strategy-proofness* is defined and characterized as a cognitively limited agent can recognize that truth-telling is a dominant strategy (Li [13]). While second-price auctions are not *obviously strategy-proof*, ascending auctions are *obviously strategy-proof*. Therefore, even if a single-valued solution is *strategy-proof*, simpler game forms associated with the solution work better.

Ex ante fairness should be also considered. A layman would say that he must have the same opportunities in the game form as others do. This suggests that procedural fairness can sometimes play out before the game form is actually executed as a participation

[12] For a summary of laboratory experiments on *strategy-proof* single-valued solutions, see Cason et al. [3].

constraint. *Ex ante fairness* guarantees that this cannot happen.[13]

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