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# The Relationship between Equilibrium Payoffs and Proposal Ratios in Bargaining Models

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**Abstract**

In this paper, we analyze a bargaining model which extends the model of Rubinstein (1982) from the viewpoint of the process of how a proposer is decided in each period. In our model, a player's probability to be a proposer depends on the history of proposers. We derive an explicit expression for the subgame perfect equilibrium (SPE) payoffs and how these SPE payoffs are related to the process in the bilateral model and the $n$-player model, respectively. In the bilateral model, we show that there is a unique SPE payoff. In the $n$-player model, although SPE payoffs may not be unique, we can see that there exists an SPE similar to that of the bilateral model.

In this paper, we also analyze the case where the discount factor is sufficiently large. In this case, we show that if the ratio of opportunities to be a proposer converges to some value, then players divide the profit according to the ratio of this convergent value. The main consequence of this result is that although the process used in our model has less regularity than a Markov process, we can derive the same result as in the model of Markov process.

JEL classification: C72; C73; C78
Keywords: Non-cooperative bargaining; Subgame perfect equilibrium; Proposal ratio; Limit payoff

## 1 Introduction

In this paper, we consider a basic non-cooperative bargaining problem in which players divide a pie of size 1. We analyze the model which extends the model of Rubinstein (1982) from the viewpoint of the process of how a proposer is decided in each period. In this model, we derive the subgame perfect equilibrium (SPE) payoffs and analyze how SPE payoffs are related to the frequency of proposal.

Recent research on non-cooperative bargaining models based on Rubinstein (1982) have used alternating offers, constant probabilities across periods or Markov process as the process of how a proposer is decided in each period (alternating offers and constant probabilities across periods

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can be considered as a special case of Markov process). For example, alternating offers model is used in Rubinstein (1982), Shaked and Sutton (1984) and Chae and Yang (1990), constant probabilities process is used in Fershtman and Seidmann (1993), Okada (1996) and Kalandrakis (2006), Markov process is used in Kalandrakis (2004) and Herings and Predtetchinski (2010).

However, in reality, the bargaining situations drastically change and the same situations rarely happen. These situations may not fit within the framework of the aforementioned papers since there are only a finite number of states in these processes. In contrast to these models, we analyze a more flexible model in the sense that the player's probability to be a proposer may depend on the history of proposers. By considering such a model, we can analyze more complex situations which can have an infinite number of states.

Mao (2017) and Mao and Zhang (2017) analyze the bilateral bargaining models which are not represented by Markov process to handle complex situations, but their procedures are deterministic from the viewpoint of the designer. The model analyzed in our paper extend their models in the sense that we consider stochastic process.

Although Merlo and Wilson (1995) and Merlo and Wilson (1998) each analyzes an $n$ player model where the probability depends on the history of proposers, the goals of these studies are to characterize the set of SPE payoffs as a fixed point of a suitable mapping. Therefore, these papers do not derive the SPE payoffs explicitly so that one do not know how players actually divide the pie other than the fact that one such division exists.

In our study, we derive an explicit expression for the SPE payoffs and how players divide the pie in the bilateral model and the $n$-player model, respectively. In the bilateral model, we show that there is a unique SPE payoff and how this SPE payoff is related to the probability to be a proposer. In the $n$-player model, although we cannot derive the uniqueness of SPE payoff, we can see that there exists an SPE which has the same form as the SPE derived in the bilateral bargaining model. Under this SPE, the relationship between players' SPE payoffs and the probability to be a proposer has the same relationship as the bilateral model.

In this paper, we also analyze the case where the discount factor is sufficiently large. This case is not analyzed in Mao (2017), Mao and Zhang (2017), Merlo and Wilson (1995) and Merlo and Wilson (1998). In the bilateral model, we show that if the ratio of opportunities to be a proposer during some periods converges to some specific value in the long run, then players divide the pie according to the ratio of this convergent value (we use the word "the proposal ratio" instead of the word "the ratio of opportunities to be a proposer" in the following). In reality, even if individuals propose the divisions freely in the beginning, the negotiation often calm down and the proposal ratio often stays in some value in the long run. Our result shows that the pie is divided according to the ratio of this value. As corollaries of this result, we can derive the results for models with alternating offers, constant probabilities across periods and Markov process. The main consequence of this result is that the process used in our model has less regularity than Markov process, we can derive the same result as in the model of Markov process. That is, the result that players divide the pie according to the ratio of the convergent value is "robust" to departures from an exact Markov process. In the $n$-player model, the same result is also derived under the SPE which has the same form as the SPE derived in the bilateral model.

This paper is organized as follows. In section 2, we define the bilateral bargaining model. In section 3, we show that there exists a unique SPE payoff in the game defined in section 2 and how SPE payoffs are related to the process of proposer. In section 4, we consider the case where the discount factor is sufficiently large in the bilateral model. In this section, we show that if the proposal ratio during some periods converges to some specific value in the long run, then players divide the pie according to the ratio of this convergent value. In section 5, we analyze the $n$-player model. In section 6, we conclude our study.

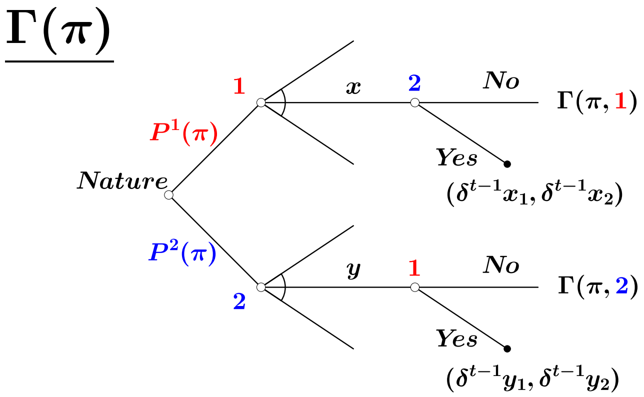


Figure 1: Subgame corresponding to $\pi$

## 2 The bilateral model

We consider the game in which two players, player 1 and 2 divide a pie of size 1. We define $N = \{1, 2\}$ as a set of players and $\delta \in (0, 1)$ as a common discount factor. Also, let $S = \{(x_1, x_2) \mid x_1 + x_2 = 1, x_1, x_2 \geq 0\}$ as the set of divisions. The game goes on as follows.

At period $t \in \{1, 2, \dots\}$, nature selects one player as a proposer. When player $i \in N$ is selected as a proposer, then she proposes some division $x \in S$. After it, the responder $j (\neq i)$ responds with Yes or No. If the responder player $j$ accepts the opponent's proposal, then the game ends and player $i$ receives $\delta^{t-1}x_i$ and player $j$ receives $\delta^{t-1}x_j$. Conversely, if player $j$ rejects the opponent's proposal, the game continues to the next period $t+1$ and repeat the above process.

In this model, we assume that the probability to be a proposer depends on the history of proposers, that is, player's probability to be a proposer depends on who proposed before the present period. Since $\bigcup_{t\in\mathbb{N}} N^{t-1}$ denotes the set of histories of proposers ($N^0 = \emptyset$), the probability that a proposer is chosen in the next period is represented by the function $P : \bigcup_{t\in\mathbb{N}} N^{t-1} \to \{(P^1, P^2) \mid P^1 + P^2 = 1, P^1, P^2 \geq 0\}$ where $P^i$ denotes player $i$'s probability.

Histories are divided into three types $H_t^a$, $H_t^b$ and $H_t^c$. $H_t^a = (N \times S \times \{No\})^{t-1}$ denotes the set of histories at the beginning of period $t$ ($H_1^a = \emptyset$), $H_t^b = H_t^a \times N$ denotes the set of histories after nature's selection and $H_t^c = H_t^b \times S$ denotes the set of histories after the proposer's offer. Let $o(h_t^a) \in N^{t-1}$ be the history of proposers in $h_t^a \in H_t^a$. Then, player $i$ is selected as a proposer with probability $P^i(o(h_t^a))$ after $h_t^a \in H_t^a$.

Consider two histories $g_t^a, h_t^a \in H_t^a$ such that $o(g_t^a) = o(h_t^a) \in N^{t-1}$. Since $P(o(g_t^a)) = P(o(h_t^a))$, the subgames corresponding to $g_t^a$ and $h_t^a$ coincide. Therefore, subgames corresponding to $H_t^a$ can be characterized by $N^{t-1}$. Now, we define $\Gamma(\pi)$ as the subgame corresponding to $\pi \in \bigcup_{t\in\mathbb{N}} N^{t-1}$ (the original game is represented by $\Gamma(\emptyset)$).

## 3 Uniqueness of SPE payoff in the bilateral model

We examine an SPE of this model. First, we prepare some notation. $\pi_r \in N^r$ denotes an order of proposers during $r$ periods and $\pi_r(k)$ denotes $k$-th proposer of the order $\pi_r$. $\pi_r^s =$

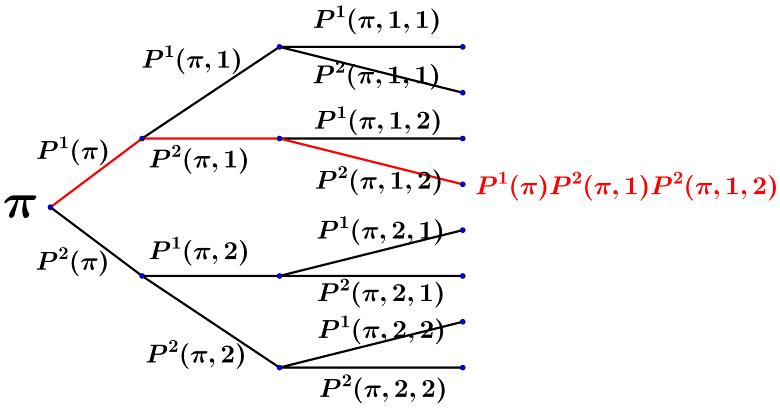


Figure 2: Extracted tree of three periods after $\pi$

$(\pi_r(1), \dots, \pi_r(s))$ denotes the proposers of the order $\pi_r$ from the first proposer to $s$-th proposer. $\pi_r\pi_s$ denotes an order of proposers in which $\pi_s$ follows $\pi_r$. Then, we define $\Pi(\pi, \pi_r) = P^{\pi_r(1)}(\pi)P^{\pi_r(2)}(\pi\pi_r^1)\cdots P^{\pi_r(r)}(\pi\pi_r^{r-1})$.

To see the meaning of $\Pi(\pi, \pi_r)$, we extract edges of nature's action from the original game tree. Figure 2 represents the tree of three periods after the history of proposer $\pi$. For example, if $\pi_3 = (1, 2, 2)$, $\Pi(\pi, \pi_3) = P^1(\pi)P^2(\pi, 1)P^2(\pi, 1, 2)$ is the probability that the red path occurs. Therefore, generally, $\Pi(\pi, \pi_r)$ is the probability that the order $\pi_r$ occurs on condition that the history of proposer $\pi$ occurred.

By the definition of $\Pi(\pi, \pi_r)$, $P^i(\pi)\Pi((\pi, i), \pi_r) = \Pi(\pi, (i, \pi_r))$ for all $\pi \in \bigcup_{t\in\mathbb{N}} N^{t-1}$, $\pi_r \in N^r$ and $i \in N$. Also, the following lemma holds.

**Lemma 1.** *For all $\pi \in \bigcup_{t\in\mathbb{N}} N^{t-1}$ and for all $r \in \mathbb{N}$, $\sum_{\pi_r\in N^r} \Pi(\pi, \pi_r) = 1$.*

*Proof.*

$$
\begin{aligned}
&\sum_{\pi_r\in N^r} \Pi(\pi,\pi_r) \\
&= \sum_{\pi_r\in N^r} P^{\pi_r(1)}(\pi)P^{\pi_r(2)}(\pi\pi_r^1)\cdots P^{\pi_r(r)}(\pi\pi_r^{r-1}) \\
&= \sum_{\pi_r\in N^r,\pi_r(r)=1} P^{\pi_r(1)}(\pi)P^{\pi_r(2)}(\pi\pi_r^1)\cdots P^{1}(\pi\pi_r^{r-1}) \\
&\quad + \sum_{\pi_r\in N^r,\pi_r(r)=2} P^{\pi_r(1)}(\pi)P^{\pi_r(2)}(\pi\pi_r^1)\cdots P^{2}(\pi\pi_r^{r-1}) \\
&= \sum_{\pi_{r-1}\in N^{r-1}} P^{\pi_{r-1}(1)}(\pi)\cdots P^1(\pi\pi_{r-1}^{r-1}) + \sum_{\pi_{r-1}\in N^{r-1}} P^{\pi_{r-1}(1)}(\pi)\cdots P^2(\pi\pi_{r-1}^{r-1}) \\
&= \sum_{\pi_{r-1}\in N^{r-1}} P^{\pi_{r-1}(1)}(\pi)\cdots P^{\pi_{r-1}(r-1)}(\pi\pi_{r-1}^{r-2})\{P^1(\pi\pi_{r-1}^{r-1}) + P^2(\pi\pi_{r-1}^{r-1})\} \\
&= \sum_{\pi_{r-1}\in N^{r-1}} \Pi(\pi,\pi_{r-1})
\end{aligned}
$$

$$
\begin{aligned}
&= \cdots \\
&= \sum_{\pi_1 \in N} \Pi(\pi, \pi_1) \\
&= P^1(\pi) + P^2(\pi) = 1.
\end{aligned}
$$

□

For all $\pi \in \bigcup_{t \in \mathbb{N}} N^{t-1}$, by using the value $\Pi(\pi, \pi_r)$, we define

$$
f_i(\pi) = \frac{\sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_{r-1} \in N^{r-1}} \Pi(\pi, \pi_{r-1}) P^i(\pi\pi_{r-1})}{\sum_{s=1}^{\infty} \delta^{s-1}}
$$

where $\Pi(\pi, \emptyset) = 1$. Now, the following lemma about $f_i(\pi)$ holds.

**Lemma 2.** *For all $\pi \in \bigcup_{t=1}^{\infty} N^{t-1}$ and $i \in N$,*

$$
f_i(\pi) = P^i(\pi)\left(1 - \delta f_j(\pi, i)\right) + P^j(\pi)\delta f_i(\pi, j)
$$

*where $j \neq i$.*

*Proof.*

$$
\begin{aligned}
& P^i(\pi)\left(1 - \delta f_j(\pi, i)\right) + P^j(\pi)\delta f_i(\pi, j) \\
=& P^i(\pi) \frac{\sum_{r=1}^{\infty} \delta^{r-1} - \delta \sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_{r-1} \in N^{r-1}} \Pi((\pi, i), \pi_{r-1}) P^j(\pi, i, \pi_{r-1})}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
& + P^j(\pi)\delta \frac{\sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_{r-1} \in N^{r-1}} \Pi((\pi, j), \pi_{r-1}) P^i(\pi, j, \pi_{r-1})}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
=& P^i(\pi) \frac{1 + \delta \sum_{r=1}^{\infty} \delta^{r-1} [1 - \sum_{\pi_{r-1} \in N^{r-1}} \Pi((\pi, i), \pi_{r-1}) P^j(\pi, i, \pi_{r-1})]}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
& + P^j(\pi)\delta \frac{\sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_{r-1} \in N^{r-1}} \Pi((\pi, j), \pi_{r-1}) P^i(\pi, j, \pi_{r-1})}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
=& P^i(\pi) \frac{1 + \delta \sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_{r-1} \in N^{r-1}} \Pi((\pi, i), \pi_{r-1}) P^i(\pi, i, \pi_{r-1})}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
& + P^j(\pi)\delta \frac{\sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_{r-1} \in N^{r-1}} \Pi((\pi, j), \pi_{r-1}) P^i(\pi, j, \pi_{r-1})}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
=& \frac{P^i(\pi) + \delta \sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_{r-1} \in N^{r-1}} P^i(\pi) \Pi((\pi, i), \pi_{r-1}) P^i(\pi, i, \pi_{r-1})}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
& + \frac{\delta \sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_{r-1} \in N^{r-1}} P^j(\pi) \Pi((\pi, j), \pi_{r-1}) P^i(\pi, j, \pi_{r-1})}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
=& \frac{P^i(\pi) + \delta \sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_{r-1} \in N^{r-1}} \Pi(\pi, (i, \pi_{r-1})) P^i(\pi, i, \pi_{r-1})}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
& + \frac{\delta \sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_{r-1} \in N^{r-1}} \Pi(\pi, (j, \pi_{r-1})) P^i(\pi, j, \pi_{r-1})}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
=& \frac{P^i(\pi) + \delta \sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_r \in N^r, \pi_r(1) = i} \Pi(\pi, \pi_r) P^i(\pi\pi_r)}{\sum_{s=1}^{\infty} \delta^{s-1}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\delta \sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_r \in N^r, \pi_r(1)=j} \Pi(\pi, \pi_r) P^i(\pi\pi_r)}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
=&\frac{P^i(\pi) + \delta \sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_r \in N^r} \Pi(\pi, \pi_r) P^i(\pi\pi_r)}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
=&f_i(\pi).
\end{aligned}
$$

□

We prove the existence of an SPE by using the above notation.

**Theorem 1.** *Consider the following pair of strategies $\sigma = (\sigma_1, \sigma_2)$. Player $i \in N$ proposes the division of the pie $(1 - \delta f_j(o(h_t^a), i), \delta f_j(o(h_t^a), i))$ to player $j(\neq i)$ at the history $(h_t^a, i) \in H_t^b$. On the other hand, player $i$ accepts player $j$'s proposal $x$ if $x_i \geq \delta f_i(o(h_t^a), j)$ and rejects if $x_i < \delta f_i(o(h_t^a), j)$ at the history $(h_t^a, j, x) \in H_t^c$. Then, $\sigma = (\sigma_1, \sigma_2)$ is an SPE of the game.*

*Proof.* Since the game is an infinite horizon discounted multi-stage game with observed actions, we can apply the one-shot deviation principle to prove Theorem 1. That is, $\sigma$ is an SPE if there is no player who can become better off by deviating from $\sigma$ for just one period (see Fudenberg and Tirole (1991)).

First, consider the history $(h_t^a, i) \in H_t^b$. If $\sigma_i$ and $\sigma_j$ are played after $(h_t^a, i)$, player $i$ receives $\delta^{t-1}(1 - \delta f_j(o(h_t^a), i))$.

Suppose that player $i$ one-shot deviates from $\sigma_i$ and proposes another division $x = (1 - x_j, x_j)$ which satisfies $x_j > \delta f_j(o(h_t^a), i)$. Then, player $j$ accepts it under $\sigma_j$ and player $i$ receives $\delta^{t-1}(1 - x_j)$. However, $\delta^{t-1}(1 - x_j)$ is smaller than $\delta^{t-1}(1 - \delta f_j(o(h_t^a), i))$. Thus, player $i$ cannot improve her payoff by proposing the division $x = (x_i, x_j)$ satisfying $x_j > \delta f_j(o(h_t^a), i)$ after the history $(h_t^a, i)$.

Next, suppose that player $i$ one-shot deviates from $\sigma_i$ and proposes another division $x = (1 - x_j, x_j)$ which satisfies $x_j < \delta f_j(o(h_t^a), i)$. Then, player $j$ rejects the offer under $\sigma_j$ and the game continues to the step after the history $(h_t^a, i, x, No) \in H_{t+1}^a$. After the history $(h_t^a, i, x, No)$, player $i$ is selected as a proposer with probability $P^i(o(h_t^a), i)$ and receives $\delta^t(1 - \delta f_j(o(h_t^a), i, i))$ under $\sigma_i$ and $\sigma_j$. On the other hand, player $i$ is selected as a responder with probability $P^j(o(h_t^a), i)$ and receives $\delta^{t+1} f_i(o(h_t^a), i, j)$ under $\sigma_i$ and $\sigma_j$. Therefore, player $i$ receives $P^i(o(h_t^a), i)\delta^t(1 - \delta f_j(o(h_t^a), i, i)) + P^j(o(h_t^a), i)\delta^{t+1} f_i(o(h_t^a), i, j) = \delta^t f_i(o(h_t^a), i)$ (by Lemma 2) after the history $(h_t^a, i, x, No)$ under $\sigma_i$ and $\sigma_j$. Now, since

$$
\begin{aligned}
&\delta^{t-1}\left(1 - \delta f_j(o(h_t^a), i)\right) - \delta^t f_i(o(h_t^a), i) \\
=&\delta^{t-1}\left[1 - \delta(f_j(o(h_t^a), i) + f_i(o(h_t^a), i))\right] \\
=&\delta^{t-1}(1 - \delta) \\
>&0,
\end{aligned}
$$

we can see

$$
\delta^{t-1}(1 - \delta f_j(o(h_t^a), i)) > \delta^t f_i(o(h_t^a), i).
$$

Therefore, player $i$ cannot improve her payoff by proposing the division $x = (x_i, x_j)$ satisfying $x_j < \delta f_j(o(h_t^a), i)$ after the history $(h_t^a, i)$.

Subsequently, we consider the subgame after the history $(h_t^a, j, x) \in H_t^c$. If player $i$ accepts the offer, she receives $\delta^{t-1} x_i$. On the other hand, if she rejects the offer, the game continues to the step after the history $(h_t^a, j, x, No) \in H_{t+1}^a$. Then, player $i$ receives $\delta^t(1 -$

$\delta f_j(o(h_t^a), j, i))$ with probability $P^i(o(h_t^a), j)$ and receives $\delta^{t+1} f_i(o(h_t^a), j, j)$ with probability $P^j(o(h_t^a), j)$ under $\sigma_i$ and $\sigma_j$. Therefore, player $i$ receives $P^i(o(h_t^a), j)\delta^t(1 - \delta f_j(o(h_t^a), j, i)) + P^j(o(h_t^a), j)\delta^{t+1} f_i(o(h_t^a), j, j) = \delta^t f_i(o(h_t^a), j)$ (by Lemma 2) after the history $(h_t^a, j, x, No)$ under $\sigma_i$ and $\sigma_j$.

Consider the case $x_i \geq \delta f_i(o(h_t^a), j)$. Suppose that player $i$ one-shot deviates from $\sigma_i$ and rejects the offer $x$. Then, the game goes to the step after the history $(h_t^a, j, x, No)$ and player $i$ receives $\delta^t f_i(o(h_t^a), j)$ under $\sigma_i$ and $\sigma_j$. In this case, we can confirm that player $i$ cannot improve her payoff by deviating from $\sigma_i$ since $\delta^{t-1} x_i \geq \delta^t f_i(o(h_t^a), j)$.

Consider the case $x_i < \delta f_i(o(h_t^a), j)$. Suppose that player $i$ one-shot deviates from $\sigma_i$ and accepts the offer $x$. Then, player $i$ receives $\delta^{t-1} x_i$. However, she can receive larger payoff $\delta^t f_i(o(h_t^a), j)$ under $\sigma_i$ and $\sigma_j$. Therefore, player $i$ cannot improve her payoff by deviating from $\sigma_i$.

Consequently, Theorem 1 holds since there is no profitable one-shot deviation. $\square$

When the SPE given in Theorem 1 is played, player 1 and 2 receive $f_1(\emptyset)$ and $f_2(\emptyset)$ (by Lemma 2), respectively. We prove that payoffs which are obtained in the SPE given in Theorem 1 are the unique SPE payoff of the game.

Let $M_i(\pi_t)$ and $m_i(\pi_t)$ be the supremum and the infimum respectively of player $i$'s SPE payoffs of the game $\Gamma(\pi_t)$ in which all payoffs are multiplied by $1/\delta^t$. We have already confirmed that there is an SPE in the game. Therefore, for all $i \in N$ and for all $\pi \in \bigcup_{t=1}^{\infty} N^{t-1}$, $M_i(\pi)$ and $m_i(\pi)$ are well-defined. We derive two inequalities involving the supremum and the infimum of SPE payoffs.

**Lemma 3.** *For all $i \in N$, for all $t \in \{1, 2, \dots\}$ and for all $h_t^a \in H_t^a$, the following inequalities hold.*

$$M_i(\pi_{t-1}) \leq P^i(\pi_{t-1}) + \delta\left[-P^i(\pi_{t-1}) m_j(\pi_{t-1}, i) + P^j(\pi_{t-1}) M_i(\pi_{t-1}, j)\right], \tag{1}$$
$$m_i(\pi_{t-1}) \geq P^i(\pi_{t-1}) + \delta\left[-P^i(\pi_{t-1}) M_j(\pi_{t-1}, i) + P^j(\pi_{t-1}) m_i(\pi_{t-1}, j)\right] \tag{2}$$

*where $\pi_{t-1} = o(h_t^a)$.*

*Proof.* Let $G(h_t^a, i)$ be the subgame after the history $(h_t^a, i) \in H_t^b$ in which all payoffs are multiplied by $1/\delta^t$. Fix $i \in N$ and $h_t^a \in H_t^a$. First, we prove (1). Consider the game $G(h_t^a, i)$. When player $i$ proposes the division $x$ which satisfies $x_j < \delta m_j(\pi_{t-1}, i)$ at the first period of $G(h_t^a, i)$, player $j$ rejects this proposal in all SPEs since player $j$ can receives a payoff of at least $\delta m_j(\pi_{t-1}, i)$ at the next period or later. Thus, if player $i$'s proposal is accepted at the first period in SPE, player $i$'s payoff is not larger than $1 - \delta m_j(\pi_{t-1}, i)$. Also, player $i$ can receive a payoff of at most $\delta M_i(\pi_{t-1}, i)$ at the next period or later. Since $M_i(\pi_{t-1}, i) + m_j(\pi_{t-1}, i) \leq 1$ by the definitions of $M_i(\pi_{t-1}, i)$ and $m_i(\pi_{t-1}, i)$, $\delta M_i(\pi_{t-1}, i) \leq \delta(1 - m_j(\pi_{t-1}, i)) < 1 - \delta m_j(\pi_{t-1}, i)$. Therefore, player $i$ can receive a payoff of at most $1 - \delta m_j(\pi_{t-1}, i)$ in the game $G(h_t^a, i)$.

Next, consider the game $G(h_t^a, j)$ ($j \neq i$). Let $M_i^*(h_t^a, j)$ be the supremum of player $i$'s SPE payoffs in the game $G(h_t^a, j)$. Now, we show $M_i^*(h_t^a, j) \leq \delta M_i(\pi_{t-1}, j)$. Suppose $M_i^*(h_t^a, j) > \delta M_i(\pi_{t-1}, j)$. Then, there is an SPE $\sigma' = (\sigma_i', \sigma_j')$ in which player $j$ proposes the division $(x_i', 1 - x_i')$ satisfying $\delta M_i(\pi_{t-1}, j) < x_i' \leq M_i^*(h_t^a, j)$ and player $i$ accept it at the first period of $G(h_t^a, j)$ since player $i$ cannot achieve a payoff larger than $\delta M_i(\pi_{t-1}, j)$ at the next period or later in all SPEs. Therefore, under $\sigma'$, player $j$ obtains $1 - x_i'$. However, player $j$ can improve her payoff by proposing the division $(x_i^*, 1 - x_i^*)$ where $x_i^*$ satisfies $x_i' > x_i^* > \delta M_i(\pi_{t-1}, j)$. This proposal is also accepted by player $i$ who follows the strategy $\sigma_i'$ since player $i$ must accept all divisions satisfying $x_i > \delta M_i(\pi_{t-1}, j)$ in all SPEs. Then, player $j$ receives a payoff $1 - x_i^*$ ($> 1 - x_i'$). Therefore, for player $j$, proposing the division $(x_i', 1 - x_i')$ is not a best response to

$\sigma_i'$. This contradicts to the fact that $\sigma'$ is an SPE of the game $G(h_t^a, j)$. Therefore, $M_i^*(h_t^a, j) \leq \delta M_i(\pi_{t-1}, j)$ holds, that is, player $i$ can receive a payoff of at most $\delta M_i(\pi_{t-1}, j)$ in the game $G(h_t^a, j)$.

Finally, consider the game $\Gamma(\pi_{t-1})$ in which all payoffs are multiplied by $1/\delta^{t-1}$. This game moves to the subgame $G(h_t^a, i)$ with probability $P^i(\pi_{t-1})$ and the game $G(h_t^a, j)$ with probability $P^j(\pi_{t-1})$. Therefore, from the above discussion, we can see

$$M_i(\pi_{t-1}) \leq P^i(\pi_{t-1})\left(1 - \delta m_j(\pi_{t-1}, i)\right) + P^j(\pi_{t-1})\delta M_i(\pi_{t-1}, j).$$

This inequality coincides with (1).

Next, we prove (2). First, consider the game $G(h_t^a, i)$. Let $m_i^*(h_t^a, i)$ be the infimum of player $i$'s SPE payoffs in the game $G(h_t^a, i)$. We show $m_i^*(h_t^a, i) \geq 1 - \delta M_j(\pi_{t-1}, i)$. Suppose $m_i^*(h_t^a, i) < 1 - \delta M_j(\pi_{t-1}, i)$. Then, there is an SPE $\sigma'' = (\sigma_i'', \sigma_j'')$ in which player $i$ obtains some payoff $x_i''$ satisfying $m_i^*(h_t^a, i) \leq x_i'' < 1 - \delta M_j(\pi_{t-1}, i)$. However, player $i$ can improve her payoff by proposing the division $(x_i^{**}, 1 - x_i^{**})$ at the first period of $G(h_t^a, i)$ where $x_i^{**}$ satisfies $1 - x_i'' > 1 - x_i^{**} > \delta M_j(\pi_{t-1}, i)$. This proposal is accepted by player $j$ who follows the strategy $\sigma_i''$ since player $j$ must accept all divisions satisfying $x_j > \delta M_j(\pi_{t-1}, i)$ in all SPEs. Then, player $i$ receives a payoff $x_i^{**}$ $(> x_i'')$. Therefore, $\sigma_i''$ is not a best response to $\sigma_j''$. This contradicts to the fact that $\sigma''$ is an SPE of the game $G(h_t^a, i)$. Therefore, $m_i^*(h_t^a, i) \geq 1 - \delta M_j(\pi_{t-1}, i)$ holds, that is, player $i$ can receive a payoff of at least $1 - \delta M_j(\pi_{t-1}, i)$ in the game $G(h_t^a, i)$.

Next, consider the game $G(h_t^a, j)$. For all SPEs, if player $j$ proposes the division $x$ which satisfies $x_i < \delta m_i(\pi_{t-1}, j)$ at the first period of $G(h_t^a, j)$, player $i$ rejects this proposal since she can receive a payoff of at least $\delta m_i(\pi_{t-1}, j)$ at the next period or later. Therefore, for all SPEs, player $i$ can obtain a payoff of at least $\delta m_i(\pi_{t-1}, j)$ in the game $G(h_t^a, j)$.

Finally, consider the game $\Gamma(\pi_{t-1})$ in which all payoffs are multiplied by $1/\delta^{t-1}$. This game moves to the subgame $G(h_t^a, i)$ with probability $P^i(\pi_{t-1})$ and the subgame $G(h_t^a, j)$ with probability $P^j(\pi_{t-1})$. Therefore, from the above discussion, we can see

$$m_i(\pi_{t-1}) \geq P^i(\pi_{t-1})(1 - \delta M_j(\pi_{t-1}, i)) + P^j(\pi_{t-1})\delta m_i(\pi_{t-1}, j).$$

This inequality coincides with (2). □

Before proving the uniqueness of SPE payoff, we provide an alternative formation of $f_i(\pi)$. Let $\rho(\pi)$ be the last responder of the order $\pi \in \bigcup_{t\in\mathbb{N}} N^{t-1}$. For all $r \in \{1, 2, \dots\}$ and for all $\pi \in \bigcup_{t=1}^{\infty} N^{t-1}$, we define $a_0^i(\pi) = P^i(\pi)$ and

$$a_r^i(\pi) = \sum_{\pi_r \in N^r} \Pi(\pi, \pi_r) P^{\rho(\pi_r)}(\pi\pi_r)\,\mathrm{sgn}_i(\rho(\pi_r))$$

where $\mathrm{sgn}_i(i) = 1$ and $\mathrm{sgn}_i(j) = -1$ $(j \neq i)$.

**Lemma 4.** *For all $\pi \in \bigcup_{t=1}^{\infty} N^{t-1}$,*

$$f_i(\pi) = \sum_{r=0}^{\infty} \delta^r a_r^i(\pi).$$

*Proof.*

$$f_i(\pi) = \frac{\sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_{r-1} \in N^{r-1}} \Pi(\pi, \pi_{r-1}) P^i(\pi\pi_{r-1})}{\sum_{s=1}^{\infty} \delta^{s-1}}$$

$$
\begin{aligned}
&=(1-\delta)\sum_{r=1}^{\infty}\delta^{r-1}\sum_{\pi_{r-1}\in N^{r-1}}\Pi(\pi,\pi_{r-1})P^i(\pi\pi_{r-1})\\
&=\sum_{r=1}^{\infty}\delta^{r-1}\sum_{\pi_{r-1}\in N^{r-1}}\Pi(\pi,\pi_{r-1})P^i(\pi\pi_{r-1})-\sum_{r=1}^{\infty}\delta^{r}\sum_{\pi_{r-1}\in N^{r-1}}\Pi(\pi,\pi_{r-1})P^i(\pi\pi_{r-1})\\
&=P^i(\pi)+\sum_{r=1}^{\infty}\delta^{r}\sum_{\pi_{r}\in N^{r}}\Pi(\pi,\pi_{r})P^i(\pi\pi_{r})-\sum_{r=1}^{\infty}\delta^{r}\sum_{\pi_{r-1}\in N^{r-1}}\Pi(\pi,\pi_{r-1})P^i(\pi\pi_{r-1})\\
&=P^i(\pi)+\sum_{r=1}^{\infty}\delta^{r}\sum_{\pi_{r-1}\in N^{r-1}}\Pi(\pi,\pi_{r-1})P^i(\pi\pi_{r-1})P^i(\pi\pi_{r-1},i)\\
&\quad+\sum_{r=1}^{\infty}\delta^{r}\sum_{\pi_{r-1}\in N^{r-1}}\Pi(\pi,\pi_{r-1})P^j(\pi\pi_{r-1})P^i(\pi\pi_{r-1},j)\\
&\quad-\sum_{r=1}^{\infty}\delta^{r}\sum_{\pi_{r-1}\in N^{r-1}}\Pi(\pi,\pi_{r-1})P^i(\pi\pi_{r-1})\\
&=P^i(\pi)-\sum_{r=1}^{\infty}\delta^{r}\sum_{\pi_{r-1}\in N^{r-1}}\Pi(\pi,\pi_{r-1})P^i(\pi\pi_{r-1})P^j(\pi\pi_{r-1},i)\\
&\quad+\sum_{r=1}^{\infty}\delta^{r}\sum_{\pi_{r-1}\in N^{r-1}}\Pi(\pi,\pi_{r-1})P^j(\pi\pi_{r-1})P^i(\pi\pi_{r-1},j)\\
&=P^i(\pi)+\sum_{r=1}^{\infty}\delta^{r}\sum_{\pi_{r}\in N^{r}}\Pi(\pi,\pi_{r})P^{\rho(\pi_r)}(\pi\pi_{r})\,\mathrm{sgn}_i(\rho(\pi_r))\\
&=\sum_{r=0}^{\infty}\delta^r a_r^i(\pi).
\end{aligned}
$$

□

By using Lemma 3 and 4, we prove the uniqueness of SPE payoff.

**Theorem 2.** *For all $t \in \{0,1,\dots\}$ and $\pi \in N^t$, $(f_1(\pi), f_2(\pi))$ are the unique SPE payoffs of the game $\Gamma(\pi)$ in which all payoffs are multiplied by $1/\delta^t$. Specially, $(f_1(\emptyset), f_2(\emptyset))$ are the unique SPE payoffs of the original game.*

*Proof.* First, we prove that for all $i \in N$ and $m \in \{1,2,\dots\}$,

$$M_i(\pi) \le \sum_{r=0}^{m-1}\delta^r a_r^i(\pi) + \delta^m \sum_{\pi_m\in N^m}\Pi(\pi,\pi_m)\mu_1(\pi\pi_m)\,\mathrm{sgn}_i(\rho(\pi_m)) \tag{3}$$

and

$$m_i(\pi) \ge \sum_{r=0}^{m-1}\delta^r a_r^i(\pi) + \delta^m \sum_{\pi_m\in N^m}\Pi(\pi,\pi_m)\mu_2(\pi\pi_m)\,\mathrm{sgn}_i(\rho(\pi_m)) \tag{4}$$

where

$$\mu_1(\pi\pi_m) = \begin{cases} m_j(\pi\pi_m) & (\pi_m(m)=i),\\ M_i(\pi\pi_m) & (\pi_m(m)=j),\end{cases}$$

and

$$\mu_2(\pi\pi_m) = \begin{cases} M_j(\pi\pi_m) & (\pi_m(m) = i), \\ m_i(\pi\pi_m) & (\pi_m(m) = j). \end{cases}$$

We prove (3) and (4) by mathematical induction. The case of $m = 1$ is obvious by Lemma 3. Suppose that (3) and (4) hold for $m = k$.

Here, we only show the inequality (3). (4) is proved similarly.

$$\begin{aligned}
& M_i(\pi) \\
\leq & \sum_{r=0}^{k-1} \delta^r a_r^i(\pi) + \delta^k \sum_{\pi_k \in N^k} \Pi(\pi, \pi_k)\mu_1(\pi\pi_k)\,\mathrm{sgn}_i(\rho(\pi_k)) \\
= & \sum_{r=0}^{k-1} \delta^r a_r^i(\pi) + \delta^k \sum_{\pi_k \in N^k, \pi_k(k)=i} \Pi(\pi, \pi_k) m_j(\pi\pi_k) \cdot (-1) + \delta^k \sum_{\pi_k \in N^k, \pi_k(k)=j} \Pi(\pi, \pi_k) M_i(\pi\pi_k) \\
\leq & \sum_{r=0}^{k-1} \delta^r a_r^i(\pi) \\
& + \delta^k \sum_{\pi_k \in N^k, \pi_k(k)=i} \Pi(\pi, \pi_k)\{P^j(\pi\pi_k) + \delta[-P^j(\pi\pi_k)M_i(\pi\pi_k, j) + P^i(\pi\pi_k)m_j(\pi\pi_k, i)]\} \cdot (-1) \\
& + \delta^k \sum_{\pi_k \in N^k, \pi_k(k)=j} \Pi(\pi, \pi_k)\{P^i(\pi\pi_k) + \delta[-P^i(\pi\pi_k)m_j(\pi\pi_k, i) + P^j(\pi\pi_k)M_i(\pi\pi_k, j)]\} \\
= & \sum_{r=0}^{k-1} \delta^r a_r^i(\pi) + \delta^k \left( \sum_{\pi_k \in N^k, \pi_k(k)=i} \Pi(\pi, \pi_k) P^j(\pi\pi_k) \cdot (-1) + \sum_{\pi_k \in N^k, \pi_k(k)=j} \Pi(\pi, \pi_k) P^i(\pi\pi_k) \right) \\
& + \delta^{k+1} \sum_{\pi_k \in N^k, \pi_k(k)=i} \Pi(\pi, \pi_k)[P^j(\pi\pi_k)M_i(\pi\pi_k, j) - P^i(\pi\pi_k)m_j(\pi\pi_k, i)] \\
& + \delta^{k+1} \sum_{\pi_k \in N^k, \pi_k(k)=j} \Pi(\pi, \pi_k)[-P^i(\pi\pi_k)m_j(\pi\pi_k, i) + P^j(\pi\pi_k)M_i(\pi\pi_k, j)] \\
= & \sum_{r=0}^{k-1} \delta^r a_r^i(\pi) + \delta^k \sum_{\pi_k \in N^k} \Pi(\pi, \pi_k) P^{\rho(\pi_k(k))}(\pi\pi_k)\,\mathrm{sgn}_i(\rho(\pi_k)) \\
& + \delta^{k+1} \sum_{\pi_k \in N^k} \Pi(\pi, \pi_k)[-P^i(\pi\pi_k)m_j(\pi\pi_k, i) + P^j(\pi\pi_k)M_i(\pi\pi_k, j)] \\
= & \sum_{r=0}^{k} \delta^r a_r^i(\pi) + \delta^{k+1} \sum_{\pi_{k+1} \in N^{k+1}} \Pi(\pi, \pi_{k+1})\mu_1(\pi\pi_{k+1})\,\mathrm{sgn}_i(\rho(\pi_{k+1})).
\end{aligned}$$

Now, we can see that (3) (and (4)) holds for $m = k+1$. Therefore, for all $i \in N$ and $m \in \{1, 2, \dots\}$,

$$M_i(\pi) \leq \sum_{r=0}^{m-1} \delta^r a_r^i(\pi) + \delta^m \sum_{\pi_m \in N^m} \Pi(\pi, \pi_m)\mu_1(\pi\pi_m)\,\mathrm{sgn}_i(\rho(\pi_m))$$

and

$$m_i(\pi) \geq \sum_{r=0}^{m-1} \delta^r a_r^i(\pi) + \delta^m \sum_{\pi_m \in N^m} \Pi(\pi, \pi_m)\mu_2(\pi\pi_m)\,\mathrm{sgn}_i(\rho(\pi_m))$$

hold.

Next, we consider taking the limit as $m \to \infty$ in (3). Focus on the second term of the right hand side in (3). Since $-1 \leq \mu_1(\pi\pi_m)\operatorname{sgn}_i(\rho(\pi_m)) \leq 1$ by the definition of $\mu_1$,

$$\begin{aligned}
-\delta^m =& -\delta^m \sum_{\pi_m \in N^m} \Pi(\pi, \pi_m) \\
\leq& \delta^m \sum_{\pi_m \in N^m} \Pi(\pi, \pi_m)\mu_1(\pi\pi_m)\operatorname{sgn}_i(\rho(\pi_m)) \\
\leq& \delta^m \sum_{\pi_m \in N^m} \Pi(\pi, \pi_m) \\
=& \delta^m
\end{aligned}$$

holds. Therefore, by taking the limit of both sides of the inequality,

$$\lim_{m\to\infty}\left(\delta^m \sum_{\pi_m \in N^m} \Pi(\pi, \pi_m)\mu_1(\pi\pi_m)\operatorname{sgn}_i(\rho(\pi_m))\right) = 0.$$

As a result, by Lemma 4,

$$\begin{aligned}
M_i(\pi) \leq& \sum_{r=0}^{\infty} \delta^r a_r^i(\pi) + \lim_{m\to\infty}\left(\delta^m \sum_{\pi_m \in N^m} \Pi(\pi, \pi_m)\mu_1(\pi\pi_m)\operatorname{sgn}_i(\rho(\pi_m))\right) \\
=& \sum_{r=0}^{\infty} \delta^r a_r^i(\pi) \\
=& f_i(\pi).
\end{aligned}$$

Similarly,

$$m_i(\pi) \geq f_i(\pi)$$

holds. Thus, since $M_i(\pi) \geq m_i(\pi)$,

$$M_i(\pi) = m_i(\pi) = f_i(\pi)$$

holds. This equation means that for all $t \in \{0, 1, \dots\}$ and $\pi \in N^t$, $(f_1(\pi), f_2(\pi))$ are the unique SPE payoffs of the game $\Gamma(\pi)$ in which all payoffs are multiplied by $1/\delta^t$. □

Finally, we provide an interpretation of the SPE payoff $f_i(\emptyset)$. $\Pi(\emptyset, \pi_{t-1})P^i(\pi_{t-1})$ roughly represents the probability that the history of proposers $(\pi_{t-1}, i) \in N^t$ occurs. Therefore, $\sum_{\pi_{t-1}\in N^{t-1}} \Pi(\emptyset, \pi_{t-1})P^i(\pi_{t-1})$ can be considered as player $i$'s probability to be a proposer at period $t$. Thus, by the first form of $f_i(\emptyset)$, we can view each component game at period $t$ involving players dividing a pie of size $\delta^{t-1}$ according to the proposal ratio at period $t$. $\delta^{t-1}\sum_{\pi_{t-1}\in N^{t-1}} \Pi(\emptyset, \pi_{t-1})P^i(\pi_{t-1})$ is the value that player $i$ can obtain at period $t$. The numerator of $f_i(\emptyset)$ is sum of these values and the denominator denotes the value of whole game. Therefore, the player with more chances to be a proposer can obtain a higher payoff.

# 4 The limit of SPE payoffs in the bilateral model

Although it is generally difficult to examine the limit of SPE payoffs in our model, if the process has some property during a time period of a certain length (which Markov process satisfies), we can give a simple expression of the limit of SPE payoffs. In this section, we define $p_t^i$ as player $i$'s probability to be a proposer at period $t$, that is,

$$p_t^i = \sum_{\pi_{t-1} \in N^{t-1}} \Pi(\emptyset, \pi_{t-1}) P^i(\pi_{t-1}).$$

**Theorem 3.** *If there exists some $k \in \mathbb{N}$ such that $\lim_{m\to\infty} \sum_{t=(m-1)k+1}^{mk} p_t^i$ converges to some value $V$ ($\lim_{m\to\infty} \sum_{t=(m-1)k+1}^{mk} p_t^j$ converges to $k-V$), then $\lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{V}{k}$ and $\lim_{\delta\uparrow 1} f_j(\emptyset) = \frac{k-V}{k}$.*

Since $p_t^i$ is player $i$'s probability to be a proposer at period $t$, $\sum_{t=(m-1)k+1}^{mk} p_t^i$ is sum of these probabilities during the periods $(m-1)k+1, \ldots, mk$. Theorem 3 means that if the proposal ratio during a time period of a certain length converges to some value, then players divide the pie according to this ratio.

*Proof.* By the assumption, for all $\epsilon > 0$, there exists some $N^* \in \mathbb{N}$ such that

$$V - \epsilon \leq \sum_{t=(m-1)k+1}^{mk} p_t^i \leq V + \epsilon$$

for $m \geq N^*$. Therefore, for $m \geq N^*$,

$$\delta^{mk-1}(V-\epsilon) \leq \sum_{t=(m-1)k+1}^{mk} \delta^{t-1} p_t^i \leq \delta^{(m-1)k}(V+\epsilon). \tag{5}$$

We define

$$\begin{aligned}
L(\delta) =& \frac{\sum_{t=1}^{(N^*-1)k} \delta^{t-1} p_t^i}{\sum_{t=1}^{\infty} \delta^{t-1}} + \frac{\sum_{m=N^*}^{\infty} \delta^{mk-1}(V-\epsilon)}{\sum_{t=1}^{\infty} \delta^{t-1}} \\
=& \frac{\sum_{t=1}^{(N^*-1)k} \delta^{t-1} p_t^i}{\sum_{t=1}^{\infty} \delta^{t-1}} + \frac{(V-\epsilon)\frac{\delta^{N^*k-1}}{1-\delta^k}}{\frac{\sum_{t=1}^{k} \delta^{t-1}}{1-\delta^k}} \\
=& \frac{\sum_{t=1}^{(N^*-1)k} \delta^{t-1} p_t^i}{\sum_{t=1}^{\infty} \delta^{t-1}} + \frac{(V-\epsilon)\delta^{N^*k-1}}{\sum_{t=1}^{k} \delta^{t-1}}
\end{aligned}$$

and

$$\begin{aligned}
R(\delta) =& \frac{\sum_{t=1}^{(N^*-1)k} \delta^{t-1} p_t^i}{\sum_{t=1}^{\infty} \delta^{t-1}} + \frac{\sum_{m=N^*}^{\infty} \delta^{(m-1)k}(V+\epsilon)}{\sum_{t=1}^{\infty} \delta^{t-1}} \\
=& \frac{\sum_{t=1}^{(N^*-1)k} \delta^{t-1} p_t^i}{\sum_{t=1}^{\infty} \delta^{t-1}} + \frac{(V+\epsilon)\frac{\delta^{(N^*-1)k}}{1-\delta^k}}{\frac{\sum_{t=1}^{k} \delta^{t-1}}{1-\delta^k}} \\
=& \frac{\sum_{t=1}^{(N^*-1)k} \delta^{t-1} p_t^i}{\sum_{t=1}^{\infty} \delta^{t-1}} + \frac{(V+\epsilon)\delta^{(N^*-1)k}}{\sum_{t=1}^{k} \delta^{t-1}}.
\end{aligned}$$

Now, $\lim_{\delta\uparrow 1} L(\delta) = \frac{V-\epsilon}{k}$ and $\lim_{\delta\uparrow 1} R(\delta) = \frac{V+\epsilon}{k}$
By the definition of $L(\delta)$, $R(\delta)$ and (5),

$$L(\delta) \leq f_i(\emptyset) \leq R(\delta)$$

holds since

$$f_i(\emptyset) = \frac{\sum_{t=1}^{(N^*-1)k} \delta^{t-1} p_t^i}{\sum_{t=1}^{\infty} \delta^{t-1}} + \frac{\sum_{m=N^*}^{\infty} \sum_{t=(m-1)k+1}^{mk} \delta^{t-1} p_t^i}{\sum_{t=1}^{\infty} \delta^{t-1}}.$$

Therefore,

$$\begin{aligned} &\frac{V-\epsilon}{k} \\ =&\lim_{\delta\uparrow 1} L(\delta) = \liminf_{\delta\uparrow 1} L(\delta) \\ \leq&\liminf_{\delta\uparrow 1} f_i(\emptyset) \leq \limsup_{\delta\uparrow 1} f_i(\emptyset) \\ \leq&\limsup_{\delta\uparrow 1} R(\delta) = \lim_{\delta\uparrow 1} R(\delta) \\ =&\frac{V+\epsilon}{k}. \end{aligned} \tag{6}$$

Since (6) holds for all $\epsilon > 0$,

$$\lim_{\delta\uparrow 1} f_i(\emptyset) = \liminf_{\delta\uparrow 1} f_i(\emptyset) = \limsup_{\delta\uparrow 1} f_i(\emptyset) = \frac{V}{k}.$$

Thus,

$$\lim_{\delta\uparrow 1} f_j(\emptyset) = 1 - \lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{k-V}{k}.$$

□

We have shown that if the proposal ratio during a time period of a certain length converges to some value, then players divide the pie according to the ratio of this value. One interpretation of the condition "there exists some $k \in \mathbb{N}$ such that $\lim_{m\to\infty} \sum_{t=(m-1)k+1}^{mk} p_t^i$ converges to some value V" is that, in reality, even if individuals propose the divisions freely in the beginning, the negotiation often calm down and the ratio of frequencies of proposal during a time period of a certain length often stays in some value in the long run. Then, the number $k$ in Theorem 3 can be considered as the length of this time period. That is, after the negotiation calm down, the ratio of individuals' opportunities to be a proposer are $V : k - V$ during these periods. Theorem 3 represents that individuals divide the pie with the ratio $V : k - V$ under this situation.

From Theorem 3, we obtain some corollaries.

**Corollary 1.** *If there exists some $k \in \mathbb{N}$ such that $\lim_{m\to\infty} \sum_{t=m+1}^{m+k} p_t^i$ converges to some value $V$, then $\lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{V}{k}$ and $\lim_{\delta\uparrow 1} f_j(\emptyset) = \frac{k-V}{k}$.*

*Proof.* If the sequence $\{\sum_{t=m+1}^{m+k} p_t^i\}_{m\in\mathbb{N}}$ converges to $V$, the subsequence $\{\sum_{t=(m-1)k+1}^{mk} p_t^i\}_{m\in\mathbb{N}}$ converges to $V$. Therefore, by Theorem 3, $\lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{V}{k}$ and $\lim_{\delta\uparrow 1} f_j(\emptyset) = \frac{k-V}{k}$. □

Corollary 1 means that the proposal ratio during $k$ consecutive periods converges to $V : k - V$, then players divide the pie according to this ratio. To help one understand, we give an example which is a generalization of the alternating offers process used in Rubinstein (1982) where $p_{2(m-1)+1}^1 = 1$ and $p_{2m}^1 = 0$ for all $m \in \mathbb{N}$.

**Example 1.** *Suppose that* $\{p^i_{2(m-1)+1}\}_{m\in\mathbb{N}}$ *converges to* $V_1$ *and* $\{p^i_{2m}\}_{m\in\mathbb{N}}$ *converges to* $V_2$*, then* $\lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{V_1+V_2}{2}$ *and* $\lim_{\delta\uparrow 1} f_j(\emptyset) = \frac{(1-V_1)+(1-V_2)}{2}$.

*Proof.* For all $\epsilon > 0$, there exists some $N^* \in \mathbb{N}$ such that

$$V_1 - \frac{\epsilon}{2} < p^i_{2(m-1)+1} < V_1 + \frac{\epsilon}{2}$$

for $m \geq N^*$. Also, there exists some $N^{**} \in \mathbb{N}$ such that

$$V_2 - \frac{\epsilon}{2} < p^i_{2m} < V_2 + \frac{\epsilon}{2}$$

for $m \geq N^{**}$.

Therefore,

$$V_1 + V_2 - \epsilon < \sum_{t=m+1}^{m+2} p^i_t < V_1 + V_2 + \epsilon$$

for $m \geq \max\{2N^*, 2N^{**}\}$.

Thus,

$$\lim_{m\to\infty} \sum_{t=m+1}^{m+2} p^i_t = V_1 + V_2.$$

Hence, by Corollary 1,

$$\lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{V_1 + V_2}{2}$$

and

$$\lim_{\delta\uparrow 1} f_j(\emptyset) = \frac{(1-V_1)+(1-V_2)}{2}.$$

□

The following Corollary 2 is also obtained by Theorem 3.

**Corollary 2.** *If* $\lim_{t\to\infty} p^i_t$ *converges to some value* $V$*, then* $\lim_{\delta\uparrow 1} f_i(\emptyset) = V$ *and* $\lim_{\delta\uparrow 1} f_j(\emptyset) = 1 - V$.

*Proof.* This is the case of $k = 1$. □

This corollary implies that SPE payoff is equal to player's probability to be a proposer in the limit.

Markov process is used in Kalandrakis (2004) and Herings and Predtetchinski (2010) where player's probability to be a proposer in each period depends on the identity of the proposer in the last period. We prove that if player's probability depends on the previous $l$ periods, then this process satisfies the condition of Corollary 2.

**Proposition 1.** *Suppose that for all* $i \in N$ *and* $\pi \in \bigcup_{t=l+1}^{\infty} N^{t-1}$*,* $P^i(\pi) > 0$ *and* $P^i(\pi)$ *depends on the previous* $l$ *periods (for* $\pi \in \bigcup_{t=1}^{l} N^{t-1}$*,* $P^i(\pi)$ *can take arbitrary values). Then, the limit* $\lim_{t\to\infty} p^i_t$ *exists.*

*Proof.* $\bigcup_{t=l+1}^{\infty} N^{t-1}$ can be divided into $2^l$ states which are characterized by the history of proposers during previous $l$ periods $(i_1, \ldots, i_l) \in N^l$ (where $i_l$ denotes the proposer in the last period). We define the set of these states as $\Theta = \{\theta_1, \ldots, \theta_{2^l}\}$. Then, for all $\pi, \pi' \in \theta_m$ $(m \in \{1, \ldots, 2^l\})$, $P^i(\pi) = P^i(\pi')$ by assumption. Therefore, for all $\pi \in \theta_m$, $P^i(\pi)$ can be expressed as a constant value $P^i(\theta_m) > 0$.

Now, define $Q_{t-1}(\theta) = \sum_{\pi_{t-1} \in N^{t-1}, \pi_{t-1} \in \theta} \Pi(\emptyset, \pi_{t-1})$ and $Q_{t-1}(\Theta) = (Q_{t-1}(\theta_1), \ldots, Q_{t-1}(\theta_{2^l}))$. Let $\theta'$ be the state corresponding to $(i_1, \ldots, i_l) \in N^l$. Also, let $\theta''$ be the state corresponding to $(1, i_1, \ldots, i_{l-1})$ and $\theta'''$ be the state corresponding to $(2, i_1, \ldots, i_{l-1})$. Then,

$$Q_t(\theta') = P^{i_l}(\theta'')Q_{t-1}(\theta'') + P^{i_l}(\theta''')Q_{t-1}(\theta'''). \tag{7}$$

Therefore, we can express

$$Q_t(\Theta) = Q_{t-1}(\Theta)A$$

where $A$ is the transition matrix satisfying (7). Under this setting, since

$$\begin{aligned} p_t^i &= \sum_{\pi_{t-1} \in N^{t-1}} \Pi(\emptyset, \pi_{t-1}) P^i(\pi_{t-1}) \\ &= \sum_{\theta \in \Theta} P^i(\theta) \sum_{\pi_{t-1} \in N^{t-1}, \pi_{t-1} \in \theta} \Pi(\emptyset, \pi_{t-1}) \\ &= \sum_{\theta \in \Theta} P^i(\theta) Q_{t-1}(\theta) \end{aligned}$$

for $t \geq l+1$, we can prove Proposition 1 by showing the limit $\lim_{t\to\infty} Q_t(\Theta)$ $\left(= \lim_{t\to\infty} Q_l(\Theta)A^{t-l}\right)$ exists. It is sufficient to show that $A$ is ergodic. We show that all entries of $A^l$ are positive, that is, show that we can arrive at any state from any state in $l$ steps with positive probability.

Let $\theta_m$ be the state corresponding to $(m_1, \ldots, m_l) \in N^l$ and $\theta_{m'}$ be the state corresponding to $(m_1', \ldots, m_l') \in N^l$. We can arrive at the state $\theta_1$ corresponding to $(m_2, \ldots, m_l, m_1') \in N^l$ from $\theta_m$ in 1 step with probability $P^{m_1'}(\theta_m) > 0$ (since for all $\pi \in \bigcup_{t=l+1}^{\infty} N^{t-1}$, $P^i(\pi) > 0$). Also, we can arrive at the state $\theta_2$ corresponding to $(m_3, \ldots, m_l, m_1', m_2') \in N^l$ from $\theta_m$ in 2 steps with probability $P^{m_1'}(\theta_m)P^{m_2'}(\theta_1) > 0$. Similarly, we can arrive at the state $\theta_{m'}$ from $\theta_m$ in $l$ steps with probability $P^{m_1'}(\theta_m)P^{m_2'}(\theta_1)\cdots P^{m_l'}(\theta_{l-1}) > 0$. Since $\theta_m$ and $\theta_{m'}$ can be taken arbitrarily, we can arrive at any state from any state in $l$ steps with positive probability. Therefore, $A$ is ergodic. □

By Proposition 1, we can see that the Markov process satisfies the condition of Corollary 2 and players divide the pie according to the proposal ratio in the limit. A special case of Corollary 2 is as follows.

**Example 2.** *Let $P^i(i) > 0$ and $P^i(j) > 0$ be constant values $(j \neq i)$. Suppose that $P^i(\pi_t) = P^i(i)$ when $\pi_t(t) = i$ and $P^i(\pi_t) = P^i(j)$ when $\pi_t(t) = j$ ($P^i(\emptyset)$ and $P^j(\emptyset)$ can take arbitrary values). Then, $\lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{P^i(j)}{P^i(j)+P^j(i)}$.*

*Proof.*

$$\begin{aligned} p_{t+1}^i &= \sum_{\pi_t \in N^t} \Pi(\emptyset, \pi_t) P^i(\pi_t) \\ &= \sum_{\pi_t \in N^t, \pi_t(t)=i} \Pi(\emptyset, \pi_t) P^i(i) + \sum_{\pi_t \in N^t, \pi_t(t)=j} \Pi(\emptyset, \pi_t) P^i(j) \end{aligned}$$

$$
\begin{aligned}
=&P^i(i)\sum_{\pi_{t-1}\in N^{t-1}}\Pi(\emptyset,\pi_{t-1})P^i(\pi_{t-1})+P^i(j)\sum_{\pi_{t-1}\in N^{t-1}}\Pi(\emptyset,\pi_{t-1})P^j(\pi_{t-1})\\
=&P^i(i)\sum_{\pi_{t-1}\in N^{t-1}}\Pi(\emptyset,\pi_{t-1})P^i(\pi_{t-1})+P^i(j)\sum_{\pi_{t-1}\in N^{t-1}}\Pi(\emptyset,\pi_{t-1})\left(1-P^i(\pi_{t-1})\right)\\
=&P^i(i)\sum_{\pi_{t-1}\in N^{t-1}}\Pi(\emptyset,\pi_{t-1})P^i(\pi_{t-1})-P^i(j)\sum_{\pi_{t-1}\in N^{t-1}}\Pi(\emptyset,\pi_{t-1})P^i(\pi_{t-1})\\
&+P^i(j)\sum_{\pi_{t-1}\in N^{t-1}}\Pi(\emptyset,\pi_{t-1})\\
=&\left(P^i(i)-P^i(j)\right)\sum_{\pi_{t-1}\in N^{t-1}}\Pi(\emptyset,\pi_{t-1})P^i(\pi_{t-1})+P^i(j)\\
=&\left(P^i(i)-P^i(j)\right)p^i_t+P^i(j).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&p^i_{t+1}-\frac{P^i(j)}{1+P^i(j)-P^i(i)}=\left(P^i(i)-P^i(j)\right)\left(p^i_t-\frac{P^i(j)}{1+P^i(j)-P^i(i)}\right)\\
\Rightarrow\quad &p^i_{t+1}-\frac{P^i(j)}{P^i(j)+P^j(i)}=\left(P^i(i)-P^i(j)\right)\left(p^i_t-\frac{P^i(j)}{P^i(j)+P^j(i)}\right).
\end{aligned}
$$

Thus,

$$
p^i_{t+1}=\frac{P^i(j)}{P^i(j)+P^j(i)}+\left(P^i(i)-P^i(j)\right)^t\left(P^i(\emptyset)-\frac{P^i(j)}{P^i(j)+P^j(i)}\right).
$$

Since $-1<P^i(i)-P^i(j)<1$,

$$
\lim_{t\to\infty}p^i_{t+1}=\frac{P^i(j)}{P^i(j)+P^j(i)}.
$$

By Corollary 2, we obtain

$$
\lim_{\delta\uparrow 1}f_i(\emptyset)=\frac{P^i(j)}{P^i(j)+P^j(i)}.
$$

□

Markov process satisfies the condition of Corollary 2 and players divide the pie according to the proposal ratio in the limit. The main consequence of Theorem 3 is that although the process used in Theorem 3 has less regularity than a Markov process, we can derive the same result as in the model of Markov process. That is, the result that players divide the pie according to the proposal ratio in the limit is "robust" to departures from an exact Markov process.

## 5 The $n$-player model

In this section, we consider the $n$-player bargaining model ($n>2$). Although SPE payoffs may not be unique in the $n$-player model if $\delta$ is large (see Merlo and Wilson (1995)), we can see that there is an SPE similar to Theorem 1. Under this SPE, we obtain the results which correspond to Theorem 3 and Proposition 1 in the $n$-player model.

## 5.1 The model

We consider the game in which $n$ players divide a pie of size 1. We redefine $N = \{1, 2, \ldots, n\}$ as a set of players and $\delta \in (0, 1)$ as a common discount factor. Also, let $S = \{ (x_1, x_2, \ldots, x_n) \mid \sum_{i \in N} x_i = 1, x_i \geq 0\}$ as the set of divisions. As with the bilateral model, we assume that a probability to be a proposer depends on the history of proposers. Since $\bigcup_{t \in \mathbb{N}} N^{t-1}$ denotes the set of histories of proposers ($N^0 = \emptyset$), the probability that a proposer is chosen in the next period is represented by the function $P : \bigcup_{t \in \mathbb{N}} N^{t-1} \to \{(P^1, P^2, \ldots, P^n) \mid \sum_{i \in N} P^i = 1, P^i \geq 0\}$ where $P^i$ denotes player $i$'s probability. The game goes on as follows.

At period $t \in \{1, 2, \ldots\}$, nature selects one player as a proposer. The player who is selected as a proposer proposes some division $x \in S$. After it, all other players respond with Yes or No sequentially (the order of responders does not affect our results). If all responders accept the proposal, then the game ends and player $i \in N$ receives $\delta^{t-1} x_i$. Conversely, if some responder rejects the proposal, the game continues to the next period $t + 1$ and repeat the above process.

## 5.2 SPE

We use the same notation as the bilateral model. That is, $\pi_r \in N^r$ denotes an order of proposers during $r$ periods and $\pi_r(k)$ denotes $k$-th proposer of the order $\pi_r$. $\pi_r^s = (\pi_r(1), \ldots, \pi_r(s))$ denotes the proposers of the order $\pi_r$ from the first proposer to $s$-th proposer. $\pi_r \pi_s$ denotes an order of proposers in which $\pi_s$ follows $\pi_r$. Also, we redefine $\Pi(\pi, \pi_r) = P^{\pi_r(1)}(\pi) P^{\pi_r(2)}(\pi \pi_r^1) \cdots P^{\pi_r(r)}(\pi \pi_r^{r-1})$ and

$$f_i(\pi) = \frac{\sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_{r-1} \in N^{r-1}} \Pi(\pi, \pi_{r-1}) P^i(\pi \pi_{r-1})}{\sum_{s=1}^{\infty} \delta^{s-1}}$$

for all $\pi \in \bigcup_{t \in \mathbb{N}} N^{t-1}$. Now, the property about $\Pi(\pi, \pi_r)$ given in Lemma 1 similarly holds in the $n$ player model. Also, the following lemma holds.

**Lemma 5.** *For all $\pi \in \bigcup_{t=1}^{\infty} N^{t-1}$ and $i \in N$,*

$$f_i(\pi) = P^i(\pi) \left( 1 - \sum_{j \neq i} \delta f_j(\pi, i) \right) + \sum_{j \neq i} P^j(\pi) \delta f_i(\pi, j).$$

*Proof.*

$$\begin{aligned}
& P^i(\pi) \left( 1 - \delta \sum_{j \neq i} f_j(\pi, i) \right) + \sum_{j \neq i} P^j(\pi) \delta f_i(\pi, j) \\
=& P^i(\pi) \frac{\sum_{r=1}^{\infty} \delta^{r-1} - \delta \sum_{j \neq i} \sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_{r-1} \in N^{r-1}} \Pi((\pi, i), \pi_{r-1}) P^j(\pi, i, \pi_{r-1})}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
& + \sum_{j \neq i} P^j(\pi) \delta \frac{\sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_{r-1} \in N^{r-1}} \Pi((\pi, j), \pi_{r-1}) P^i(\pi, j, \pi_{r-1})}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
=& P^i(\pi) \frac{1 + \delta \sum_{r=1}^{\infty} \delta^{r-1} [1 - \sum_{j \neq i} \sum_{\pi_{r-1} \in N^{r-1}} \Pi((\pi, i), \pi_{r-1}) P^j(\pi, i, \pi_{r-1})]}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
& + \sum_{j \neq i} P^j(\pi) \delta \frac{\sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_{r-1} \in N^{r-1}} \Pi((\pi, j), \pi_{r-1}) P^i(\pi, j, \pi_{r-1})}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
=& P^i(\pi) \frac{1 + \delta \sum_{r=1}^{\infty} \delta^{r-1} \sum_{\pi_{r-1} \in N^{r-1}} \Pi((\pi, i), \pi_{r-1}) P^i(\pi, i, \pi_{r-1})}{\sum_{s=1}^{\infty} \delta^{s-1}}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j\neq i} P^j(\pi)\delta\frac{\sum_{r=1}^{\infty}\delta^{r-1}\sum_{\pi_{r-1}\in N^{r-1}}\Pi((\pi,j),\pi_{r-1})P^i(\pi,j,\pi_{r-1})}{\sum_{s=1}^{\infty}\delta^{s-1}}\\
=&\frac{P^i(\pi)+\delta\sum_{r=1}^{\infty}\delta^{r-1}\sum_{\pi_{r-1}\in N^{r-1}}P^i(\pi)\Pi((\pi,i),\pi_{r-1})P^i(\pi,i,\pi_{r-1})}{\sum_{s=1}^{\infty}\delta^{s-1}}\\
&+\frac{\delta\sum_{r=1}^{\infty}\delta^{r-1}\sum_{j\neq i}\sum_{\pi_{r-1}\in N^{r-1}}P^j(\pi)\Pi((\pi,j),\pi_{r-1})P^i(\pi,j,\pi_{r-1})}{\sum_{s=1}^{\infty}\delta^{s-1}}\\
=&\frac{P^i(\pi)+\delta\sum_{r=1}^{\infty}\delta^{r-1}\sum_{\pi_{r-1}\in N^{r-1}}\Pi(\pi,(i,\pi_{r-1}))P^i(\pi,i,\pi_{r-1})}{\sum_{s=1}^{\infty}\delta^{s-1}}\\
&+\frac{\delta\sum_{r=1}^{\infty}\delta^{r-1}\sum_{j\neq i}\sum_{\pi_{r-1}\in N^{r-1}}\Pi(\pi,(j,\pi_{r-1}))P^i(\pi,j,\pi_{r-1})}{\sum_{s=1}^{\infty}\delta^{s-1}}\\
=&\frac{P^i(\pi)+\delta\sum_{r=1}^{\infty}\delta^{r-1}\sum_{\pi_r\in N^r,\pi_r(1)=i}\Pi(\pi,\pi_r)P^i(\pi,\pi_r)}{\sum_{s=1}^{\infty}\delta^{s-1}}\\
&+\frac{\delta\sum_{r=1}^{\infty}\delta^{r-1}\sum_{j\neq i}\sum_{\pi_r\in N^r,\pi_r(1)=j}\Pi(\pi,\pi_r)P^i(\pi,\pi_r)}{\sum_{s=1}^{\infty}\delta^{s-1}}\\
=&\frac{P^i(\pi)+\delta\sum_{r=1}^{\infty}\delta^{r-1}\sum_{\pi_r\in N^r}\Pi(\pi,\pi_r)P^i(\pi,\pi_r)}{\sum_{s=1}^{\infty}\delta^{s-1}}\\
=&\frac{P^i(\pi)+\sum_{r=1}^{\infty}\delta^{r}\sum_{\pi_r\in N^r}\Pi(\pi,\pi_r)P^i(\pi,\pi_r)}{\sum_{s=1}^{\infty}\delta^{s-1}}\\
=&f_i(\pi).
\end{aligned}
$$

□

In the $n$ player model, there exists the following SPE.

**Theorem 4.** *Consider the following strategies $\sigma = (\sigma_1, \ldots, \sigma_n)$. After the history of proposers $\pi \in \bigcup_{t=1}^{\infty} N^{t-1}$, if player $i \in N$ becomes a proposer, she proposes the division $x^* \in S$ where $x_j^* = \delta f_j(\pi, i)$ for $j \neq i$ and $x_i^* = 1 - \delta \sum_{j\neq i} f_j(\pi, i)$. Conversely, when player $i$ becomes a responder, she accepts player $j$'s proposal $x \in S$ if $x_i \geq \delta f_i(\pi, j)$ and rejects if $x_i < \delta f_i(\pi, j)$. Then, $\sigma$ is an SPE of the $n$ player model.*

*Proof.* We apply the one-shot deviation principle. Consider the path after the history of proposers $\pi_{t-1} \in N^{t-1}$.

First, consider the case that player $i$ is selected as a proposer after $\pi_{t-1}$. If $\sigma$ is played, player $i$ receives $\delta^{t-1}(1 - \delta \sum_{j\neq i} f_j(\pi_{t-1}, i))$. Suppose that player $i$ one-shot deviates from $\sigma_i$ and proposes another division $x$ which satisfies $x_j \geq \delta f_j(\pi_{t-1}, i)$ for all $j \neq i$ and $x_{j'} > \delta f_{j'}(\pi_{t-1}, i)$ for some $j' \neq i$. Then, all responders accept it under $\sigma$ and player $i$ receives $\delta^{t-1}(1 - \sum_{j\neq i} x_j)$. However, $\delta^{t-1}(1 - \sum_{j\neq i} x_j)$ is smaller than $\delta^{t-1}(1 - \delta \sum_{j\neq i} f_j(\pi_{t-1}, i))$. Thus, player $i$ cannot improve her payoff by proposing the division $x$.

Next, suppose that player $i$ one-shot deviates from $\sigma_i$ and proposes another division $x'$ which satisfies $x'_{j^*} < \delta f_j(\pi_{t-1}, i)$ for some $j^* \neq i$. Then, player $j^*$ rejects the offer under $\sigma_{j^*}$ and the game continues to the next period. Then, the history of proposers is $(\pi_{t-1}, i)$. After this history, player $i$ is selected as a proposer with probability $P^i(\pi_{t-1}, i)$ and receives $\delta^t(1 - \delta \sum_{j\neq i} f_j(\pi_{t-1}, i, i))$ under $\sigma$. On the other hand, player $j \neq i$ is selected as a proposer with probability $P^j(\pi_{t-1}, i)$ and player $i$ receives $\delta^{t+1} f_i(\pi_{t-1}, i, j)$ under $\sigma$. Therefore, player $i$

receives $P^i(\pi_{t-1}, i)\delta^t(1-\delta\sum_{j\neq i} f_j(\pi_{t-1}, i, i)) + \sum_{j\neq i} P^j(\pi_{t-1}, i)\delta^{t+1} f_i(\pi_{t-1}, i, j) = \delta^t f_i(\pi_{t-1}, i)$ (by Lemma 5) under $\sigma$. Now, since

$$\begin{aligned}
&\delta^{t-1}\left(1-\delta\sum_{j\neq i} f_j(\pi_{t-1}, i)\right) - \delta^t f_i(\pi_{t-1}, i) \\
=&\delta^{t-1}\left(1-\delta\sum_{j'\in N} f_{j'}(\pi_{t-1}, i)\right) \\
=&\delta^{t-1}(1-\delta) \\
>&0,
\end{aligned}$$

we can see

$$\delta^{t-1}\left(1-\delta\sum_{j\neq i} f_j(\pi_{t-1}, i)\right) > \delta^t f_i(\pi_{t-1}, i).$$

Therefore, player $i$ cannot improve her payoff by proposing the division $x'$.

Subsequently, consider the case that player $j^{**} \neq i$ is selected as a proposer after $\pi_{t-1}$ and she proposes the division $x \in S$. If player $i$ accepts the offer, she receives $\delta^{t-1}x_i$. On the other hand, if she rejects the offer, the game continues to the next period. Then, player $i$ receives $\delta^t(1-\delta\sum_{j\neq i} f_j(\pi_{t-1}, j^{**}, i))$ with probability $P^i(\pi_{t-1}, j^{**})$ and receives $\delta^{t+1} f_i(\pi_{t-1}, j^{**}, j)$ with probability $P^j(\pi_{t-1}, j^{**})$ for $j \neq i$ under $\sigma$. Therefore, if player $i$ rejects player $j^{**}$'s proposal $x$, she receives $P^i(\pi_{t-1}, j^{**})\delta^t(1-\delta\sum_{j\neq i} f_j(\pi_{t-1}, j^{**}, i)) + \sum_{j\neq i} P^j(\pi_{t-1}, j^{**})\delta^{t+1} f_i(\pi_{t-1}, j^{**}, j) = \delta^t f_i(\pi_{t-1}, j^{**})$ (by Lemma 5) under $\sigma$.

Consider the case $x_i \geq \delta f_i(\pi_{t-1}, j^{**})$. Suppose that player $i$ one-shot deviates from $\sigma_i$ and rejects the offer $x$. Then, the game continues to the next period and player $i$ receives $\delta^t f_i(\pi_{t-1}, j^{**})$ under $\sigma$. In this case, we can confirm that player $i$ cannot improve her payoff by deviating from $\sigma_i$ since $\delta^{t-1}x_i \geq \delta^t f_i(\pi_{t-1}, j^{**})$.

Consider the case $x_i < \delta f_i(\pi_{t-1}, j^{**})$. Suppose that player $i$ one-shot deviates from $\sigma_i$ and accepts the offer $x$. Then, player $i$ receives $\delta^{t-1}x_i$. However, she can receive larger payoff $\delta^t f_i(\pi_{t-1}, j^{**})$ under $\sigma$. Therefore, player $i$ cannot improve her payoff by deviating from $\sigma_i$.

Consequently, Theorem 4 holds since there is no profitable one-shot deviation. □

## 5.3 The limit of the SPE payoff

If the SPE $\sigma = (\sigma_1, \ldots, \sigma_n)$ given in Theorem 4 is played, player $i \in N$ receives the payoff $f_i(\emptyset)$. Under this SPE, we obtain the results which correspond to Theorem 3 and Proposition 1 in the $n$ player model.

We redefine

$$p_t^i = \sum_{\pi_{t-1}\in N^{t-1}} \Pi(\emptyset, \pi_{t-1}) P^i(\pi_{t-1}).$$

**Theorem 5.** *If there exists some $k \in \mathbb{N}$ such that $\{(\sum_{t=(m-1)k+1}^{mk} p_t^1, \ldots, \sum_{t=(m-1)k+1}^{mk} p_t^n)\}_{m\in\mathbb{N}}$ converges to some values $(V_1, \ldots, V_n)$, then $\lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{V_i}{k}$ for all $i \in N$.*

*Proof.* The proof is the same as Theorem 3. □

**Corollary 3.** *If there exists some $k \in \mathbb{N}$ such that $\{(\sum_{t=m+1}^{m+k} p_t^1, \ldots, \sum_{t=m+1}^{m+k} p_t^n)\}_{m\in\mathbb{N}}$ converges to some values $(V_1, \ldots, V_n)$, then $\lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{V_i}{k}$ for all $i \in N$.*

**Corollary 4.** *If $\{(p_t^1, \ldots, p_t^n)\}_{t\in\mathbb{N}}$ converges to some values $(V_1, \ldots, V_n)$, then $\lim_{\delta\uparrow 1} f_i(\emptyset) = V_i$ for all $i \in N$.*

Therefore, players divide the pie according to the proposal ratio under the SPE given in Theorem 4.

If player's probability to be a proposer depends on the previous $l$ periods (Markov process), this process satisfies the condition of Corollary 4.

**Proposition 2.** *Suppose that for all $i \in N$ and $\pi \in \bigcup_{t=l+1}^{\infty} N^{t-1}$, $P^i(\pi) > 0$ and $P^i(\pi)$ depends only on the previous $l$ periods (for $\pi \in \bigcup_{t=1}^{l} N^{t-1}$, $P^i(\pi)$ can take arbitrary values). Then, $\{(p_t^1, \ldots, p_t^n)\}_{t\in\mathbb{N}}$ converges.*

*Proof.* $\bigcup_{t=l+1}^{\infty} N^{t-1}$ can be divided into $n^l$ states which are characterized by the history of proposers during previous $l$ periods. The rest of the proof is the same as Proposition 1. □

Therefore, in the $n$ player model, under the SPE $\sigma = (\sigma_1, \ldots, \sigma_n)$, we obtain the same results as the bilateral model.

# 6 Conclusion

We analyzed the model which extends the model of Rubinstein (1982) from the viewpoint of the process of how a proposer is decided in each period. In the bilateral bargaining model, we derived the unique SPE payoffs and analyzed how the SPE payoffs are related to the process. We saw each component game at period $t$ involving players dividing a pie of size $\delta^{t-1}$ according to the proposal ratio at period $t$ in the unique SPE payoffs. Therefore, the player with more chances to be a proposer can obtain a higher payoff.

In the case $\delta \to 1$, we showed if the proposal ratio ultimately converges to some value, then players divide the pie according to this convergent value. The main consequence of Theorem 3 is that although the process used in Theorem 3 has less regularity than a Markov process, we can derive the same result as in the model of Markov process.

In the $n$-player model, we showed that there exists an SPE which has the same form as the SPE given in Theorem 1. Under this SPE, we showed that the same results as the bilateral model hold.

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