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Comparison of the voluntary contribution and Pareto-efficient mechanisms under voluntary participation*

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Abstract

We compare the voluntary contribution mechanism with any mechanism attaining Pareto-efficient allocations when each agent can choose whether she participates in the mechanism for the provision of a non-excludable public good. In our participation game, the equilibrium participation probability of each agent under the voluntary contribution mechanism becomes greater than that under any Pareto-efficient mechanism as the number of agents increases. Moreover, both the equilibrium expected provision level of the public good and the equilibrium expected payoff of each agent under the voluntary contribution mechanism may be higher than those under any Pareto-efficient mechanism.

Keywords: voluntary participation; non-excludable public goods; voluntary contribution mechanism; Pareto-efficient mechanism; mixed strategies

JEL codes: C72; D71; D78; H41

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1 Introduction

This paper considers the provision of non-excludable public goods. It is well known that when public goods are provided in a decentralized fashion, the so-called free-rider problem arises (Samuelson, 1954). Following Groves and Ledyard (1977), who were the first to propose a mechanism whose Nash equilibrium allocations are Pareto-efficient, a vast body of literature has been published on the resolution of the free-rider problem.[1]

Besides the free-rider problem, another incentive problem arises in the provision of non-excludable public goods, called the "participation problem": an agent may have an incentive not to participate in the mechanism because she can obtain benefits from public goods that are provided by participants due to non-excludability (Olson, 1965). However, Groves and Ledyard (1977) and subsequent studies implicitly assumed that all agents must participate in the mechanism that the mechanism designer proposes. Therefore, these studies did not solve the participation problem.

Subsequently, several studies have examined the participation problem (Palfrey and Rosenthal, 1984; Saijo and Yamato, 1997, 1999, 2010; Dixit and Olson, 2000). To address this problem, these studies considered the following two-stage game: in the first stage, each agent simultaneously decides to "participate" or "not participate" in a given mechanism and, in the second stage, after knowing the other agents' participation decisions, the agents who chose "participation" in the first stage play the mechanism. They all derived impossibility results that everyone does not necessarily participate in the mechanism.[2]

Motivated by these impossibility results of full participation, we take a "second-best" approach. That is, we identify which mechanism has the highest rate of participation in a given class of mechanisms. To this end, we compare mechanisms based on their rates of participation. To the best of our knowledge, there has been no study comparing mechanisms in the same environment where each agent has the freedom of

[1]The main drawback of the Groves-Ledyard mechanism is that it is not individual rational, that is, the equilibrium allocation determined by the mechanism does not necessarily satisfy the condition where it is at least as good as each agent's initial endowment. Hurwicz (1979) and Walker (1981) subsequently succeeded in constructing a mechanism whose Nash equilibrium allocations are Lindahl allocations that satisfy both Pareto-efficiency and individual rationality. After that, numerous mechanisms that satisfy additional desirable properties, such as individual feasibility and balancedness, have been proposed. See, for example, Groves and Ledyard (1987), Tian (1990), Hurwicz (1994), Dutta et al. (1995), and Suzuki (2009).

[2]Specifically, the aforementioned studies mainly considered either the mechanism attaining Pareto-efficient allocations or the voluntary contribution mechanism. Saijo and Yamato (1997, 2010) showed that these impossibility results hold for a more general class of mechanisms in the economy where the amount of the public good is continuous.

non-participation.[3]

As a first step in comparing several public provision mechanisms, this paper restricts its attention to the two types of mechanisms. First, we consider any mechanism in normal or extensive form attaining Pareto-efficient allocations, which we call a *Pareto-efficient mechanism*.[4] Second, we consider the *voluntary contribution mechanism*, which has been studied by many authors, although it cannot realize Pareto-efficient allocations.[5] Following Saijo and Yamato (1997, 1999), we also consider a two-stage game and assume that all agents have the same Cobb-Douglas preferences. However, unlike Saijo and Yamato (1997, 1999) who focused on pure strategies, this paper examines a symmetric mixed strategy Nash equilibrium of the first stage (hereafter, participation game). Since the agents are homogenous in our model, it is reasonable to focus on symmetric mixed strategy Nash equilibria. Another reason for this focus is that coordination on any asymmetric equilibrium would be difficult. In fact, some studies only focused on symmetric mixed strategy Nash equilibria due to coordination difficulties (Bagnoli and Lipman, 1988; Holmström and Nalebuff, 1992).

We first show that there is a unique symmetric mixed strategy Nash equilibrium in the participation game under both the voluntary contribution mechanism and any Pareto-efficient mechanism. Next, we find that the probability of participation in the symmetric mixed strategy Nash equilibrium decreases as the number of agents in an economy increases under both mechanisms. Finally, we numerically compare the voluntary contribution mechanism with any Pareto-efficient mechanism from the viewpoints of participation probabilities, expected provision levels of the public good, and expected payoffs. The equilibrium participation probability of each agent under the voluntary contribution mechanism becomes greater than that under any Pareto-efficient mechanism as the number of agents in an economy increases. Moreover, both the equilibrium expected provision level of the public good and the equilibrium expected payoff of each agent under the voluntary contribution mechanism become higher than those under

[3]Saijo and Yamato (1997, 1999, 2010) considered multiple mechanisms, but did not compare them. On the other hand, Palfrey and Rosenthal (1984) and Dixit and Olson (2000) considered one specific mechanism. Recently, there is a growing literature that examines the participation problem (Koriyama, 2009; Shinohara, 2009, 2015; Healy, 2010; Furusawa and Konishi, 2011; Matsushima and Shinohara, 2012; Konishi and Shinohara, 2014). These studies also focused on specific mechanisms.

[4]Under the aforementioned mechanisms, agents choose strategies simultaneously. These are called mechanisms in normal form. On the other hand, under the Pareto-efficient mechanisms for public goods proposed by Moore and Repullo (1988) and Varian (1994), agents select strategies sequentially. These are called mechanisms in extensive form.

[5]Warr (1982, 1983) and Bergstrom et al. (1986) theoretically investigated the properties of the voluntary contribution mechanism. See Ledyard (1995) and Chen (2008) for surveys on the experimental results of the voluntary contribution mechanism.

any Pareto-efficient mechanism when the number of agents and the value of the public good are sufficiently large. Our results suggest that the voluntary contribution mechanism might be superior to any Pareto-efficient mechanism if we allow agents to choose participation in the mechanism voluntarily.

Some studies are closely related to ours. Palfrey and Rosenthal (1984) and Dixit and Olson (2000) examined symmetric mixed strategy Nash equilibria in their participation games. They considered the problem of the provision of a binary public good, while the amount of the public good is continuous in this paper. Palfrey and Rosenthal (1984) considered the voluntary contribution mechanism both with and without a refund in the second stage. In the second stage of Dixit and Olson (2000), participants play a cooperative game of Coasian bargaining to determine whether to provide the public good. Okada (1993) and Hong and Karp (2012) also investigated a symmetric mixed strategy Nash equilibrium of a similar participation problem for n-person prisoners' dilemmas and international environmental agreements, respectively. They also showed that all agents do not participate and the equilibrium participation probability decreases as the number of agents increases in their settings.[6]

The rest of this paper is organized as follows. Section 2 introduces the model, mechanisms, and participation game. Section 3 presents examples to illustrate our basic idea. Section 4 investigates symmetric mixed strategy Nash equilibria in the participation game under the voluntary contribution mechanism and those under any Pareto-efficient mechanism. Section 5 numerically compares the voluntary contribution mechanism with any Pareto-efficient mechanism. Section 6 provides concluding remarks. All proofs are relegated to the appendices.

2 Preliminaries

2.1 Model

We consider the following economies with one private good and one pure public good. Let $N = \{1, 2, \dots, n\}$ be the set of agents, with generic element i. Agent i's consumption bundle is denoted by $(x_i, y) \in \mathbb{R}^2_+$, where x_i is the level of private good she consumes on her own and y is the level of the public good. Each agent $i \in N$ has a

[6] Dixit and Olson (2000) derived these results using simulation analysis. Hong and Lim (2016) provided analytical and experimental results that support Dixit and Olson's (2000) simulation results. Koriyama (2009) also examined a symmetric mixed strategy Nash equilibrium for the provision of a binary public good. However, he did not investigate the relationship between the number of agents in an economy and the equilibrium participation probability.

preference relation represented by a (symmetric) Cobb-Douglas utility function on her consumption space $\mathbb{R}^2_+$: for each $(x_i, y) \in \mathbb{R}^2_+$, $u_i^\alpha(x_i, y) = x_i^\alpha y^{1-\alpha}$, where $\alpha \in \,]0,1[$.[7] Then, the coefficient α on the private good can be identified with a utility function. Hence, the set of symmetric Cobb-Douglas utility functions is represented by the open interval $]0,1[$.

Agent i's initial endowment is denoted by $(\omega_i, 0)$, where $\omega_i > 0$. That is, there is no public good initially. In what follows, we assume that for each $i \in N$, $\omega_i = \omega > 0$. However, the public good can be produced from the private good by means of a constant returns to scale technology, and let $y = \sum_{i \in N}(\omega_i - x_i)$ be the production function of the public good. Given a non-empty set $T \subseteq N$ of agents, a *feasible allocation for* T is denoted by $(x_T, y) \equiv ((x_i)_{i \in T}, y) \in \mathbb{R}^{\#T+1}_+$ with $\sum_{i \in T}(\omega_i - x_i) = y$.[8] The set of feasible allocations for $T \subseteq N$ is denoted by A^T.

2.2 Mechanisms

A *mechanism* is a function Γ that associates with each non-empty set $T \subseteq N$ a pair $\Gamma(T) = ((M_j^T)_{j \in T}, g^T)$, where M_i^T is the *strategy space* of agent $i \in T$ and $g^T \colon \prod_{j \in T} M_j^T \to \mathbb{R}^{\#T+1}_+$ is an *outcome function* when the agents in T play the mechanism.[9] For convenience, we use notation Γ^T instead of $\Gamma(T)$. Given $g^T(m) = (x_T, y)$, let $g_i^T(m) = (x_i, y)$ for each $i \in T$ and $g_y^T(m) = y$. For each $\alpha \in \,]0,1[$ and each non-empty $T \subseteq N$, let $\mathbf{NE}(\Gamma^T, \alpha) \subseteq A^T$ denote the set of (pure strategy) Nash equilibrium allocations of Γ^T at α.

This paper considers a well-known mechanism called the voluntary contribution mechanism. Formally, the *voluntary contribution mechanism* is a mechanism such that for each non-empty $T \subseteq N$ and each $i \in T$, $M_i^T = [0, \omega_i]$ and for each $m \in \prod_{j \in T} M_j^T$, $g_i^T(m) = \left(\omega_i - m_i, \sum_{j \in T} m_j\right)$. Under the voluntary contribution mechanism, each agent $i \in T$ chooses her contribution out of her endowment to the provision of the public good, m_i, to maximize her utility $u_i^\alpha\left(\omega_i - m_i, \sum_{j \in T} m_j\right)$, given contributions $(m_j)_{j \in T \setminus \{i\}}$ of the other agents in T. We consider the Nash equilibrium allocations of the voluntary contribution mechanism.

In addition to the voluntary contribution mechanism, we consider any mechanism satisfying the following two conditions:

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[7]Let $a, b \in \mathbb{R}$ be such that $a \leq b$. Then, we denote by $[a, b]$ and $]a, b[$ the closed interval from a to b and the open interval from a to b, respectively.

[8]Given a non-empty set X, we denote by $\#X$ the cardinality of X.

[9]For simplicity, we confine our attention to mechanisms in normal form. This restriction does not affect the results.

- **Symmetry:** For each non-empty $T \subseteq N$ and each $\alpha \in]0,1[$, if for each pair $\{i,j\} \subseteq T$, $\omega_i = \omega_j$ and $(x_T, y) \in \mathbf{NE}(\Gamma^T, \alpha)$, then for each pair $\{i,j\} \subseteq T$, $x_i = x_j$.

- **Pareto-efficiency only for participants:** For each non-empty $T \subseteq N$ and each $\alpha \in]0,1[$,

$$\mathbf{NE}(\Gamma^T, \alpha) \subseteq \left\{ (x_T, y) \in A^T : \begin{array}{l} \text{there is no } (x'_T, y') \in A^T \text{ such that} \\ \text{for each } i \in T,\ u_i^\alpha(x'_T, y') \geq u_i^\alpha(x_T, y) \text{ and} \\ \text{for some } j \in T,\ u_j^\alpha(x'_T, y') > u_j^\alpha(x_T, y) \end{array} \right\}.$$

The condition of symmetry requires that if all participants have the same preferences and endowments, they receive the same consumption bundle at any Nash equilibrium. Therefore, every participant pays the same amount of the private good for the provision of the public good at any Nash equilibrium. The condition of Pareto-efficiency only for participants means that every Nash equilibrium allocation of the mechanism should be Pareto-efficient for participants, but not necessarily with respect to all agents. We call a mechanism satisfying these two conditions a *Pareto-efficient mechanism*.

Remark. Note that a well-known Lindahl mechanism is Pareto-efficient. Given a value $\alpha \in]0,1[$ and a non-empty set $T \subseteq N$ of agents, a feasible allocation (x_T, y) for T is a *Lindahl allocation for T at α* if there is a price vector $q \in \mathbb{R}_+^{\#T}$ such that for each $i \in T$, $x_i + q_i y = \omega_i$ and for each $(x'_i, y') \in \mathbb{R}_+^2$ with $x'_i + q_i y' \leq \omega_i$, $u_i^\alpha(x_i, y) \geq u_i^\alpha(x'_i, y')$. Let $\mathscr{L}^T(\alpha)$ be the set of Lindahl allocations for T at α. A *Lindahl mechanism* is a mechanism such that for each $\alpha \in]0,1[$ and each non-empty $T \subseteq N$, $\mathbf{NE}(\Gamma^T, \alpha) = \mathscr{L}^T(\alpha)$. That is, a Lindahl mechanism is a mechanism whose Nash equilibrium allocations coincide with the Lindahl allocations for each value $\alpha \in]0,1[$ and each non-empty set $T \subseteq N$ of agents. $\diamondsuit$

2.3 Participation game

Given a mechanism, we consider the following two-stage game with voluntary participation. In the first stage, each agent simultaneously decides whether she participates in the mechanism. In the second stage, knowing the other agents' participation decisions, the agents who participated in the first stage choose their strategies for the mechanism.

We derive a symmetric subgame perfect equilibrium of the two-stage game with voluntary participation. Saijo and Yamato (1999) identified a unique Nash equilibrium allocation for each possible first stage outcome in any Pareto-efficient mechanism and the voluntary contribution mechanism.

Proposition 1 (Saijo and Yamato, 1999). *Let* $\alpha \in]0,1[$ *and* $T \subseteq N$ *with* $T \neq \emptyset$. *Then:*

(i) *There exists a unique symmetric pure strategy Nash equilibrium allocation of any Pareto-efficient mechanism for* T *at* α *given by, for each* $i \in T$,

$$\left(x_i^T, y^T\right) = \left(\omega\alpha, \omega(1-\alpha)\#T\right).$$

(ii) *There exists a unique symmetric pure strategy Nash equilibrium allocation of the voluntary contribution mechanism for* T *at* α *given by, for each* $i \in T$,

$$\left(x_i^T, y^T\right) = \left(\frac{\omega\alpha\#T}{1+\alpha(\#T-1)}, \frac{\omega(1-\alpha)\#T}{1+\alpha(\#T-1)}\right).$$

Given a Nash equilibrium allocation of the second stage, the first stage can be reduced to the following simultaneous game. Given a mechanism Γ, a *participation game under* Γ is represented by $(N, (\{\mathsf{P}, \mathsf{NP}\}, (\pi_i^\Gamma))_{i\in N})$, where $\{\mathsf{P}, \mathsf{NP}\}$ is the strategy set common to all agents and π_i^Γ is agent i's payoff function. Each agent chooses either P (participation) or NP (non-participation), simultaneously. Let $T(s)$ be the set of participants at $s \equiv (s_i)_{i\in N} \in \{\mathsf{P}, \mathsf{NP}\}^{\#N}$, that is, $T(s) \equiv \{i \in N \colon s_i = \mathsf{P}\}$. The payoff of agent i is: for each $s \in \{\mathsf{P}, \mathsf{NP}\}^{\#N}$,

$$\pi_i^\Gamma(s) \equiv \begin{cases} u_i^\alpha\left(x_i^{T(s)}, y^{T(s)}\right) & \text{if } i \in T(s) \\ u_i^\alpha\left(\omega, y^{T(s)}\right) & \text{if } i \notin T(s), \end{cases}$$

where $\left((x_j^{T(s)})_{j\in T(s)}, y^{T(s)}\right) \in \mathbf{NE}(\Gamma^{T(s)}, \alpha)$.

Then, from Proposition 1, the following hold:

- If Γ is a Pareto-efficient mechanism, then for each $i \in N$ and each $s \in \{\mathsf{P}, \mathsf{NP}\}^{\#N}$,

$$\pi_i^\Gamma(s) = \begin{cases} \omega\alpha^\alpha[(1-\alpha)\#T(s)]^{1-\alpha} & \text{if } i \in T(s) \\ \omega[(1-\alpha)\#T(s)]^{1-\alpha} & \text{if } i \notin T(s). \end{cases}$$

- If Γ is the voluntary contribution mechanism, then for each $i \in N$ and for each

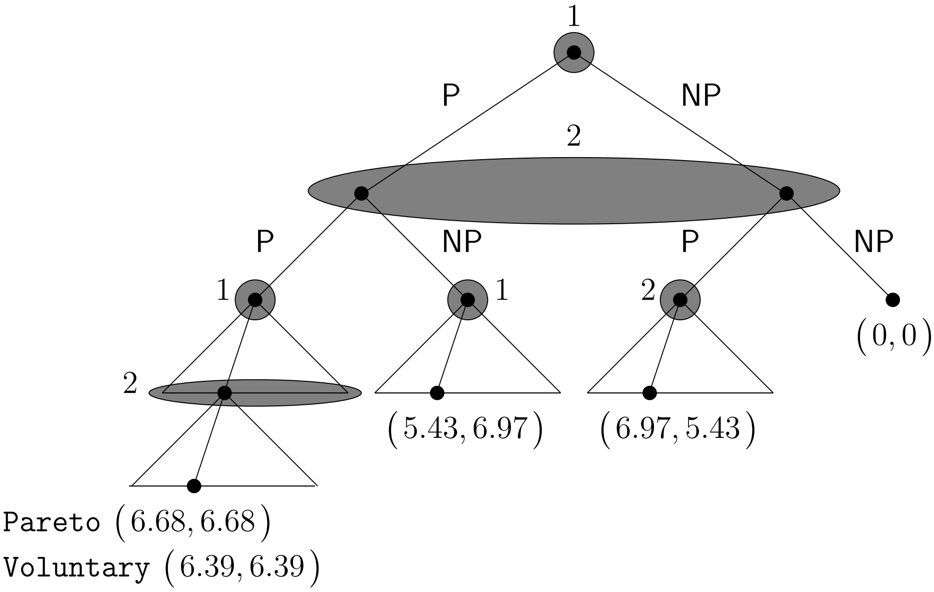

Figure 1. Game tree when agents can choose their participation to a mechanism.

$s \in \{\mathsf{P}, \mathsf{NP}\}^{\#N}$,

$$\pi_i^\Gamma(s) = \begin{cases} \dfrac{\omega\alpha^\alpha(1-\alpha)^{1-\alpha}\#T(s)}{1+\alpha(\#T(s)-1)} & \text{if } i \in T(s) \\ \dfrac{\omega[(1-\alpha)\#T(s)]^{1-\alpha}}{[1+\alpha(\#T(s)-1)]^{1-\alpha}} & \text{if } i \notin T(s). \end{cases}$$

3 Examples

This section provides examples to illustrate our basic idea. Let $\alpha = 0.7$ and $\omega = 10$.

3.1 Two-agent case

Pareto-efficient mechanism. Figure 1 illustrates the two-stage voluntary participation game under any Pareto-efficient mechanism for a two-agent case. There are three possible cases in the second stage:

1. If both agents choose P (participation), then the Nash equilibrium allocation is a unique symmetric Pareto-efficient allocation for the two-agent economy, given by $(x_1^N, x_2^N, y^N) = (7, 7, 6)$. In this case, the payoff of each participant $i \in \{1, 2\}$ is $u_i^{0.7}(x_i^N, y^N) \approx 6.68$.

2. If agent i selects P, but the other agent $j \neq i$ chooses NP (non-participation), then only participant i plays the mechanism. In this case, the Nash equilibrium

	P (p)	NP $(1-p)$
P (p)	6.68, 6.68	5.43, 6.97
NP $(1-p)$	6.97, 5.43	0, 0

Figure 2. Payoff matrix of any Pareto-efficient mechanism when agents can choose their participation.

allocation is a unique Pareto-efficient allocation for the economy of only one participant i, given by $(x_i^{\{i\}}, y^{\{i\}}) = (7, 3)$. Then, the payoff of participant i is $u_i^{0.7}(x_i^{\{i\}}, y^{\{i\}}) \approx 5.43$, whereas that of non-participant j is $u_j^{0.7}(\omega, y^{\{i\}}) \approx 6.97$. Note that non-participant j can enjoy the non-excludable public good produced by participant i, $y^{\{i\}}$, although she makes no contribution to the provision of the public good.

3. If both select NP, no public good is produced. In this case, the payoff of each non-participant $i \in \{1, 2\}$ is $u_i^{0.7}(\omega, 0) = 0$.

Figure 2 shows the payoff matrix for the first stage decision on participation, in which the second stage equilibrium payoff is indicated for each possible case. We now derive a unique symmetric mixed strategy Nash equilibrium in Figure 2. Let $p \in [0, 1]$ be the probability of participation that each agent chooses under a symmetric mixed strategy profile. Then, each agent's expected payoff of choosing P when the other agent chooses P with probability p under any Pareto-efficient mechanism is

$$p \times 6.68 + (1-p) \times 5.43,$$

and her expected payoff of choosing NP is

$$p \times 6.97 + (1-p) \times 0.$$

We denote these two expected payoffs as $U_{\mathsf{P}}^{\mathrm{PE}}(p)$ and $U_{\mathsf{NP}}^{\mathrm{PE}}(p)$. At a non-degenerate mixed strategy equilibrium, $U_{\mathsf{P}}^{\mathrm{PE}}(p) = U_{\mathsf{NP}}^{\mathrm{PE}}(p)$. Therefore, $U_{\mathsf{P}}^{\mathrm{PE}}(p) - U_{\mathsf{NP}}^{\mathrm{PE}}(p) = -(6.97 - 6.68)p + 5.43(1-p) = 0$, which we can alternatively write as

$$0.29p = 5.43(1-p). \tag{1}$$

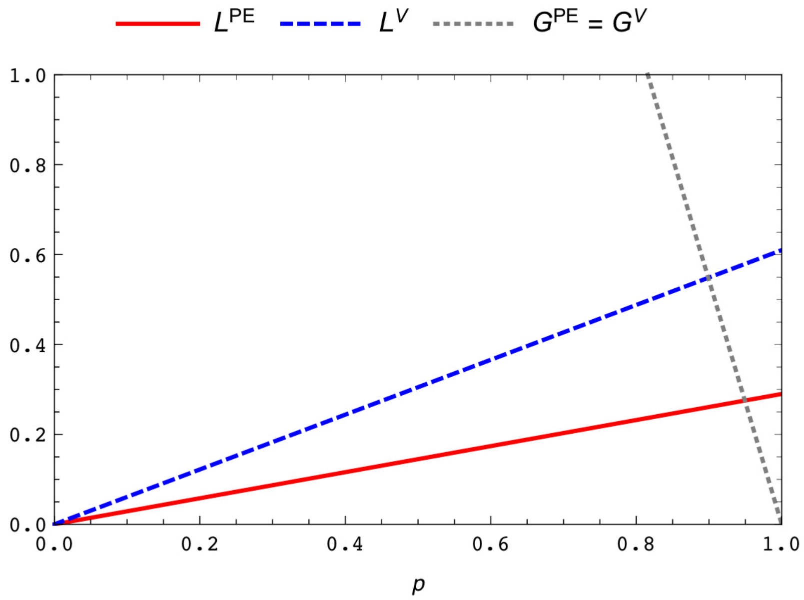

Figure 3. Graphs of L^{PE}, L^{V}, and $G^{\mathrm{PE}} = G^{\mathrm{V}}$.

The left-hand side of (1), $L^{\mathrm{PE}}(p) \equiv 0.29p$, represents the expected payoff loss of choosing P when the other agent selects P with probability p under any Pareto-efficient mechanism. On the other hand, the right-hand side of (1), $G^{\mathrm{PE}}(p) \equiv 5.43(1-p)$, denotes the expected payoff gain of choosing P when the other agent selects NP with probability $(1-p)$ under any Pareto-efficient mechanism. The expected payoff loss L^{PE} is increasing in p, the participation probability of the other agent, while the expected payoff gain G^{PE} is decreasing in p (see Figure 3). At equilibrium, the expected payoff loss L^{PE} is equal to the expected payoff gain G^{PE}. There is a unique mixed strategy equilibrium given by 0.9502.[10]

Voluntary contribution mechanism. Similarly to Pareto-efficient mechanisms, we can examine the two-stage voluntary participation game under the voluntary contribution mechanism. Figure 1 illustrates the game tree and Figure 4 shows the payoff matrix for the first stage decision on participation in the voluntary contribution mechanism, in which the second stage equilibrium payoffs are denoted for each possible case.[11] Then, each agent's expected payoff of choosing P when the other agent chooses

[10] Incidentally, in Figure 2, there are two pure Nash equilibria, in which one agent chooses P, whereas the other agent does not. This paper focuses on the symmetric mixed-strategy equilibrium.

[11] It is easy to see that the Nash equilibrium allocations of the voluntary contribution mechanism are given by $(x_1^N, x_2^N, y^N) = (\frac{140}{17}, \frac{140}{17}, \frac{60}{17})$ when both agents choose P and $(x_i^{\{i\}}, y^{\{i\}}) = (7, 3)$ when only agent i selects P.

	P (p)	NP ($1-p$)
P (p)	6.39, 6.39	5.43, 6.97
NP ($1-p$)	6.97, 5.43	0, 0

Figure 4. Payoff matrix of the voluntary contribution mechanism when agents can choose their participation.

P with probability p under the voluntary contribution mechanism is

$$p \times 6.36 + (1-p) \times 5.43,$$

and the expected payoff of choosing NP is

$$p \times 6.97 + (1-p) \times 0.$$

We denote these two expected payoffs as $U^{\mathrm{V}}_{\mathsf{P}}(p)$ and $U^{\mathrm{V}}_{\mathsf{NP}}(p)$. At a non-degenerate mixed strategy equilibrium, $U^{\mathrm{V}}_{\mathsf{P}}(p) = U^{\mathrm{V}}_{\mathsf{NP}}(p)$. Therefore, $U^{\mathrm{V}}_{\mathsf{P}}(p) - U^{\mathrm{V}}_{\mathsf{NP}}(p) = -(6.97-6.36)p + 5.43(1-p) = 0$, that is,

$$0.61p = 5.43(1-p). \tag{2}$$

The left-hand side of (2), $L^{\mathrm{V}}(p) \equiv 0.61p$, represents the expected payoff loss of choosing P when the other agent selects P with probability p under the voluntary contribution mechanism. On the other hand, the right-hand side of (2), $G^{\mathrm{V}}(p) \equiv 5.43(1-p)$, denotes the expected gain of choosing P when the other agent selects NP with probability $(1-p)$ under the voluntary contribution mechanism. The expected payoff loss L^{V} is increasing in p, while the expected payoff gain G^{V} is decreasing in p (see Figure 3). There is a unique mixed strategy equilibrium, at which the expected payoff loss L^{V} equals the expected payoff gain G^{V}, given by 0.9032.[12]

Comparison between two mechanisms. Note that the equilibrium participation probability under any Pareto-efficient mechanism, 0.9502, is greater than that under the voluntary contribution mechanism, 0.9032. As per Figure 3, the expected payoff loss under any Pareto-efficient mechanism, $L^{\mathrm{PE}}(p)$, is lower than that under the voluntary contribution mechanism, $L^{\mathrm{V}}(p)$, for any positive value of p and both are increasing

[12]Incidentally, in Figure 4, there are two pure Nash equilibria, in which one agent chooses P, whereas the other agent does not.

	P (p)	NP $(1-p)$
P (p)	7.55, 7.55, 7.55	6.68, 8.58, 6.68
NP $(1-p)$	8.58, 6.68, 6.68	6.97, 6.97, 5.43

P (p)

	P (p)	NP $(1-p)$
P (p)	6.68, 6.68, 8.58	5.43, 6.97, 6.97
NP $(1-p)$	6.97, 5.43, 6.97	0, 0, 0

NP $(1-p)$

Figure 5. Payoff matrix of any Pareto-efficient mechanism when agents can choose their participation. Agent 1 chooses one of the two rows, agent 2 chooses one of the two columns, and agent 3 chooses one of the two matrices.

in p. In addition, the expected payoff gains under the two mechanisms are the same, $G^{\text{PE}} = G^{\text{V}}$, and are decreasing in p. Hence, the equilibrium participation probability under any Pareto-efficient mechanism at which L^{PE} intersects G^{PE} is greater than that under the voluntary contribution mechanism at which L^{V} intersects G^{V}.

3.2 Three-agent case

As shown above, the equilibrium participation probability under any Pareto-efficient mechanism is greater than that under the voluntary contribution mechanism when there are two agents. However, when there are more than two agents, the equilibrium participation probability under any Pareto-efficient mechanism may be less than or equal to that under the voluntary contribution mechanism.

Pareto-efficient mechanism. One can easily verify that the Pareto-efficient allocations of any Pareto-efficient mechanism are given by $(x_1^N, x_2^N, x_3^N, y^N) = (7,7,7,9)$ when three agents choose P; $(x_i^{\{i,j\}}, x_j^{\{i,j\}}, y^{\{i,j\}}) = (7,7,6)$ when two agents, i and j, select P; and $(x_i^{\{i\}}, y^{\{i\}}) = (7,3)$ when only one agent i chooses P. Figure 5 shows the payoff matrix of any Pareto-efficient mechanism for the first stage decision on participation, in which the second equilibrium payoffs are denoted for each of the possible eight cases. We derive a unique symmetric mixed strategy Nash equilibrium in Figure 5. Let $p \in [0,1]$ be the probability of participation that each agent chooses in a symmetric mixed strategy profile. At a non-degenerate mixed strategy equilibrium, each agent's expected payoff of choosing P when the other agent selects P with probability p under any Pareto-efficient mechanism,

$$U_{\mathsf{P}}^{\text{PE}}(p) = p^2 \times 7.55 + 2 \times p(1-p) \times 6.68 + (1-p)^2 \times 5.43,$$

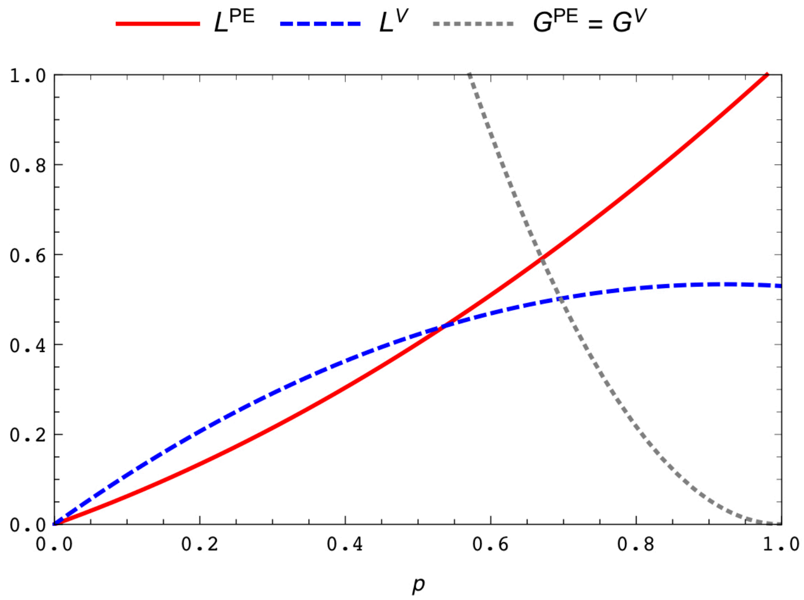

Figure 6. Graphs of L^{PE}, L^{V}, and $G^{\mathrm{PE}} = G^{\mathrm{V}}$.

should be equal to her expected payoff of choosing NP,

$$U_{\mathsf{NP}}^{\mathrm{PE}}(p) = p^2 \times 8.58 + 2 \times p(1-p) \times 6.97 + (1-p)^2 \times 0.$$

Therefore, $U_{\mathsf{P}}^{\mathrm{PE}}(p) - U_{\mathsf{NP}}^{\mathrm{PE}}(p) = -(8.58-7.55)p^2 - 2 \times (6.97-6.68)p(1-p) + 5.43(1-p)^2 = 0$, that is,

$$1.03p^2 + 0.58p(1-p) = 5.43(1-p)^2. \tag{3}$$

The left-hand side of (3), $L^{\mathrm{PE}}(p) \equiv 1.03p^2 + 0.58p(1-p)$, consists of two terms:

- The first term, $1.03p^2$, represents the expected payoff loss of choosing P when the other two agents select P with probability p under any Pareto-efficient mechanism.
- The second term, $0.58p(1-p)$, expresses another expected payoff loss of choosing P when one of the other two agents selects P with probability p and one of them chooses NP with probability $1-p$ under any Pareto-efficient mechanism.

On the other hand, the right-hand side of (3), $G^{\mathrm{PE}}(p) \equiv 5.43(1-p)^2$, denotes the expected payoff gain of choosing P when the other two agents select NP with probability $1-p$ under any Pareto-efficient mechanism (see Figure 6). There is a unique mixed strategy equilibrium, at which the expected payoff loss L^{PE} equals the expected payoff

	P (p)	NP ($1-p$)
P (p)	6.79, 6.79, 6.79	6.39, 7.32, 6.39
NP ($1-p$)	7.32, 6.39 , 6.39	6.97, 6.97, 5.43

P (p)

	P (p)	NP ($1-p$)
P (p)	6.39, 6.39, 7.32	5.43, 6.97, 6.97
NP ($1-p$)	6.97, 5.43, 6.97	0, 0, 0

NP ($1-p$)

Figure 7. Payoff matrix of the voluntary contribution mechanism when agents can choose their participation. Agent 1 chooses one of the two rows, agent 2 chooses one of the two columns, and agent 3 chooses one of the two matrices.

gain G^{PE}, given by 0.6705.[13]

Voluntary contribution mechanism. It is easy to see that the Nash equilibrium allocations of the voluntary contribution mechanism are given by $(x_1^N, x_2^N, x_3^N, y^N) = (8.75, 8.75, 8.75, 3.75)$ when three agents choose P; $(x_i^{\{i,j\}}, x_j^{\{i,j\}}, y^{\{i,j\}}) = (\frac{140}{17}, \frac{140}{17}, \frac{60}{17})$ when two agents, i and j, select P; and $(x_i^{\{i\}}, y^{\{i\}}) = (7, 3)$ when only one agent i chooses P. Figure 7 shows the payoff matrix for the first stage decision on participation in the voluntary contribution mechanism with three agents. At a non-degenerate mixed strategy equilibrium, each agent's expected payoff of choosing P when the other agents select P with probability p under the voluntary contribution mechanism,

$$U_{\mathsf{P}}^{\mathrm{V}}(p) = p^2 \times 6.79 + 2 \times p(1-p) \times 6.39 + (1-p)^2 \times 5.43,$$

should be equal to her expected payoff of choosing NP,

$$U_{\mathsf{NP}}^{\mathrm{V}}(p) = p^2 \times 7.32 + 2 \times p(1-p) \times 6.97 + (1-p)^2 \times 0.$$

Therefore, $U_{\mathsf{P}}^{\mathrm{V}}(p) - U_{\mathsf{NP}}^{\mathrm{V}}(p) = -(7.32-6.79)p^2 - 2\times(6.97-6.39)p(1-p) + 5.43(1-p)^2 = 0$, that is,

$$0.53p^2 + 1.16p(1-p) = 5.43(1-p)^2. \tag{4}$$

The left-hand side of (4), $L^{\mathrm{V}}(p) \equiv 0.53p^2 + 1.16p(1-p)$, consists of two terms:

- The first term, $0.53p^2$, represents the expected payoff loss of choosing P when the other two agents select P with probability p under the voluntary contribution mechanism.

[13] Incidentally, in Figure 5, there are three pure Nash equilibria under which only one agent chooses P.

- The second term, $1.16p(1-p)$, expresses another expected payoff loss of choosing P when one of the other two agents selects P with probability p and the other chooses NP with probability $1-p$ under the voluntary contribution mechanism.

On the other hand, the right-hand side of (4), $G^{\mathrm{V}}(p) \equiv 5.43(1-p)^2$, denotes the expected payoff gain of choosing P when the other two agents select NP with probability $1-p$ under the voluntary contribution mechanism (see Figure 6). There is a unique mixed strategy equilibrium, at which the expected payoff loss L^{V} equals the expected payoff gain G^{V}, given by 0.6956.[14]

Comparison between two mechanisms. Note that the equilibrium participation probability under the voluntary contribution mechanism, 0.6956, is greater than that under any Pareto-efficient mechanism, 0.6705. As per Figure 6, the expected payoff loss under any Pareto-efficient mechanism, $L^{\mathrm{PE}}(p)$, is higher than that under the voluntary contribution mechanism, $L^{\mathrm{V}}(p)$, for a sufficiently large value of p. Moreover, the expected payoff gains under the two mechanisms are the same, $G^{\mathrm{PE}} = G^{\mathrm{V}}$, and decreasing in p. Therefore, the equilibrium participation probability under any Pareto-efficient mechanism at which L^{PE} intersects G^{PE} is lower than that under the voluntary contribution mechanism at which L^{V} intersects G^{V}. Intuitively, the incentive of deviating from participation under any Pareto-efficient mechanism is higher than that under the voluntary contribution mechanism. This is because the provision level of the public good under any Pareto-efficient mechanism is higher than that under the voluntary contribution mechanism, which leads to a greater equilibrium participation probability under the voluntary contribution mechanism than that under any Pareto-efficient mechanism.

Given this observation, it is natural to ask the following question. How often is the equilibrium participation probability under the voluntary contribution mechanism greater than that under any Pareto-efficient mechanism? In fact, if there are at least three agents, the equilibrium participation probability under the voluntary contribution mechanism is greater than that under any Pareto-efficient mechanism unless both equilibrium participation probabilities are equal to one. Interestingly, both the expected equilibrium payoff and the expected equilibrium provision level of the public good under the voluntary contribution mechanism become higher than those under any Pareto-efficient mechanism when the number of agents and the value of the public good are sufficiently large. We explain these facts in Section 5.

[14] Incidentally, in Figure 7, there are three pure Nash equilibria under which only one agent chooses P.

4 Symmetric mixed strategy Nash equilibrium

This section derives a symmetric mixed strategy Nash equilibrium in the participation game under any Pareto-efficient mechanism and the voluntary contribution mechanism. In the symmetric mixed strategy Nash equilibrium, all agents randomize P (participation) and NP (non-participation) with the same probability. Note that this mixed strategy is an evolutionarily stable strategy (Maynard Smith, 1982).

4.1 Pareto-efficient mechanism

We here consider any Pareto-efficient mechanism. Let $p \in [0,1]$ be the probability of participation under a symmetric mixed strategy profile. Given $\alpha \in]0,1[$ and $n \geq 2$, let $U_{\mathsf{P}}^{\mathrm{PE}}(p,\alpha,n)$ be each agent's expected payoff of choosing P when the other agents choose P with probability p under any Pareto-efficient mechanism. Explicitly,

$$U_{\mathsf{P}}^{\mathrm{PE}}(p,\alpha,n) \equiv \omega(1-\alpha)^{1-\alpha}\alpha^{\alpha}\sum_{k=0}^{n-1}\binom{n-1}{k}p^k(1-p)^{n-1-k}(k+1)^{1-\alpha},$$

where $\binom{r}{t} \equiv \frac{r!}{t!(r-t)!}$ is the binomial coefficient. Similarly, let $U_{\mathsf{NP}}^{\mathrm{PE}}(p,\alpha,n)$ be each agent's expected payoff of choosing NP when the other agents choose P with probability p under any Pareto-efficient mechanism. Explicitly,

$$U_{\mathsf{NP}}^{\mathrm{PE}}(p,\alpha,n) \equiv \omega(1-\alpha)^{1-\alpha}\sum_{k=0}^{n-1}\binom{n-1}{k}p^k(1-p)^{n-1-k}k^{1-\alpha}.$$

At a non-degenerate mixed strategy Nash equilibrium, the expected payoff of choosing P should be equal to that of choosing NP.

We now formally prove the existence and uniqueness of a symmetric mixed strategy Nash equilibrium in the participation game under any Pareto-efficient mechanism. Moreover, we show that as the number of agents increases, the equilibrium participation probability decreases and then converges to 0 as the number of agents goes to infinity.

Theorem 1. *Let $\alpha \in]0,1[$ and $n \geq 2$. Then:*

(i) *There is a unique symmetric mixed strategy Nash equilibrium $p^{\mathrm{PE}}(\alpha,n)$ in the participation game under any Pareto-efficient mechanism.*

(ii) *If $p^{\mathrm{PE}}(\alpha,n+1) < 1$, then $p^{\mathrm{PE}}(\alpha,n) > p^{\mathrm{PE}}(\alpha,n+1)$. Moreover, $\lim_{k\to\infty} p^{\mathrm{PE}}(\alpha,k) = 0$.*

	α								
n	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2	1.0000	1.0000	1.0000	1.0000	1.0000	0.9623	0.9502	0.9553	0.9731
3	1.0000	1.0000	0.7885	0.6965	0.6589	0.6529	0.6705	0.7108	0.7820
4	1.0000	0.6981	0.5558	0.4970	0.4760	0.4785	0.5005	0.5440	0.6229
5	0.8611	0.5336	0.4278	0.3849	0.3710	0.3758	0.3970	0.4375	0.5123
6	0.6925	0.4316	0.3475	0.3137	0.3036	0.3090	0.3284	0.3651	0.4336
7	0.5792	0.3622	0.2924	0.2647	0.2568	0.2622	0.2798	0.3129	0.3754
8	0.4977	0.3121	0.2524	0.2288	0.2225	0.2277	0.2437	0.2737	0.3307
9	0.4363	0.2741	0.2220	0.2015	0.1962	0.2011	0.2158	0.2431	0.2954
10	0.3884	0.2443	0.1981	0.1800	0.1755	0.1801	0.1936	0.2186	0.2668
20	0.1852	0.1171	0.0954	0.0871	0.0853	0.0880	0.0953	0.1088	0.1353
30	0.1216	0.0770	0.0628	0.0574	0.0563	0.0582	0.0632	0.0724	0.0906
40	0.0905	0.0574	0.0468	0.0428	0.0420	0.0435	0.0473	0.0543	0.0681
50	0.0720	0.0457	0.0373	0.0341	0.0335	0.0347	0.0378	0.0434	0.0545
100	0.0357	0.0227	0.0185	0.0170	0.0167	0.0173	0.0188	0.0217	0.0273
500	0.0071	0.0045	0.0037	0.0034	0.0033	0.0034	0.0038	0.0043	0.0055

Table 1. Numerical results: the equilibrium participation probability under any Pareto-efficient mechanism, $p^{\mathrm{PE}}(\alpha, n)$.

The proof of Theorem 1 is provided in Appendix A. Table 1 illustrates that the equilibrium participation probability under any Pareto-efficient mechanism decreases as the number of agents increases, given each value $\alpha \in \{0.1, \ldots, 0.9\}$.

4.2 Voluntary contribution mechanism

We next consider the voluntary contribution mechanism. Let $p \in [0,1]$ be the probability of participation under a symmetric mixed strategy profile. Given $\alpha \in]0,1[$ and $n \geq 2$, let $U^{\mathrm{V}}_{\mathsf{P}}(p, \alpha, n)$ be each agent's expected payoff of choosing P when the other agents choose P with probability p under the voluntary contribution mechanism. Explicitly,

$$U^{\mathrm{V}}_{\mathsf{P}}(p, \alpha, n) \equiv \omega(1-\alpha)^{1-\alpha}\alpha^{\alpha} \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} \frac{k+1}{1+k\alpha}.$$

Similarly, let $U^{\mathrm{V}}_{\mathsf{NP}}(p, \alpha, n)$ be each agent's expected payoff of choosing NP when the other agents choose P with probability p under the voluntary contribution mechanism. Explicitly,

$$U^{\mathrm{V}}_{\mathsf{NP}}(p, \alpha, n) \equiv \omega(1-\alpha)^{1-\alpha} \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} \left[\frac{k}{1+\alpha(k-1)}\right]^{1-\alpha}.$$

	α								
n	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2	1.0000	1.0000	1.0000	0.9861	0.9252	0.9020	0.9032	0.9222	0.9553
3	1.0000	1.0000	0.8533	0.7346	0.6876	0.6783	0.6956	0.7371	0.8089
4	1.0000	0.8921	0.6579	0.5678	0.5339	0.5315	0.5527	0.5982	0.6795
5	1.0000	0.7183	0.5322	0.4602	0.4343	0.4348	0.4561	0.5002	0.5806
6	1.0000	0.6005	0.4460	0.3863	0.3654	0.3672	0.3875	0.4287	0.5050
7	0.9714	0.5156	0.3836	0.3326	0.3151	0.3176	0.3365	0.3747	0.4462
8	0.8494	0.4517	0.3363	0.2919	0.2769	0.2797	0.2973	0.3325	0.3993
9	0.7545	0.4017	0.2994	0.2600	0.2470	0.2498	0.2662	0.2988	0.3611
10	0.6787	0.3617	0.2697	0.2344	0.2228	0.2257	0.2409	0.2713	0.3295
20	0.3382	0.1810	0.1353	0.1179	0.1125	0.1146	0.1234	0.1409	0.1751
30	0.2252	0.1206	0.0903	0.0787	0.0752	0.0768	0.0829	0.0951	0.1191
40	0.1688	0.0905	0.0677	0.0591	0.0565	0.0577	0.0624	0.0717	0.0902
50	0.1350	0.0724	0.0542	0.0473	0.0452	0.0462	0.0501	0.0576	0.0726
100	0.0674	0.0362	0.0271	0.0237	0.0227	0.0232	0.0251	0.0290	0.0367
500	0.0135	0.0072	0.0054	0.0047	0.0045	0.0047	0.0051	0.0058	0.0074

Table 2. Numerical results: the equilibrium participation probability under the voluntary contribution mechanism, $p^{\mathrm{V}}(\alpha, n)$.

At a non-degenerate mixed strategy Nash equilibrium, the expected payoff of choosing P should be equal to that of choosing NP.

We then obtain the same existence and uniqueness result as in Theorem 1. Similarly to the Pareto-efficient mechanism, the equilibrium participation probability under the voluntary contribution mechanism is monotonically decreasing in the number of agents and then converges to 0 as the number of agents goes to infinity.

Theorem 2. *Let $\alpha \in \,]0,1[$ and $n \geq 2$. Then:*

(i) *There is a unique symmetric mixed strategy Nash equilibrium $p^{\mathrm{V}}(\alpha, n)$ in the participation game under the voluntary contribution mechanism.*

(ii) *If $p^{\mathrm{V}}(\alpha, n+1) < 1$, then $p^{\mathrm{V}}(\alpha, n) > p^{\mathrm{V}}(\alpha, n+1)$. Moreover, $\lim_{k\to\infty} p^{\mathrm{V}}(\alpha, k) = 0$.*

The proof of Theorem 2 is provided in Appendix B. Table 2 illustrates that the equilibrium participation probability under the voluntary contribution mechanism decreases as the number of agents increases, given each value $\alpha \in \{0.1, \ldots, 0.9\}$.

5 Numerical comparison

This section presents the results of our numerical comparison of the voluntary contribution mechanism with any Pareto-efficient mechanism from the viewpoints of participation probabilities, expected provision levels of a public good, and expected payoffs, respectively.

5.1 Equilibrium participation probability

In Section 3, we showed that for two agents, the equilibrium participation probability under the voluntary contribution mechanism is lower than that under any Pareto-efficient mechanism using the specific values of α. In fact, this result holds for any values of α. The proof of the following theorem is provided in Appendix C.

Theorem 3. *For each* $\alpha \in]0,1[$, $p^{\mathrm{V}}(\alpha,2) \leq p^{\mathrm{PE}}(\alpha,2)$ *and for some* $\alpha \in]0,1[$, $p^{\mathrm{V}}(\alpha,2) < p^{\mathrm{PE}}(\alpha,2)$.

However, as mentioned in Section 3, this is no longer true for three or more agents. To see this, we introduce the following notion. Given $\alpha \in]0,1[$ and $n \geq 2$, let

$$PP(\alpha,n) \equiv \frac{p^{\mathrm{V}}(\alpha,n)}{p^{\mathrm{PE}}(\alpha,n)}$$

be the ratio of the equilibrium participation probability under the voluntary contribution mechanism to that under any Pareto-efficient mechanism for (α,n). We call this the *participation probability ratio.*

Table 3 reports the participation probability ratio $PP(\alpha,n)$ when α varies from 0.1 to 0.9 and n from 2 to 500. Figure 8 shows the graphs of $PP(\alpha,n)$ when $n = 2$, 3, 10, 20, 50 and α varies from 0 to 1. Both Table 3 and Figure 8 reveal that whenever $n \geq 3$ and $p^{\mathrm{V}}(\alpha,n) < 1$, $PP(\alpha,n) > 1$ (in Table 3, $PP(\alpha,n) > 1$ is highlighted in gray), that is, the equilibrium participation probability under the voluntary contribution mechanism is greater than that under any Pareto-efficient mechanism. Moreover, we observe from Table 3 and Figure 8 that the region of α for which $PP(\alpha,n) > 1$ increases as the number of agents increases.

5.2 Equilibrium expected provision level of a public good

Next, we numerically compute the equilibrium expected provision levels of a public good under the voluntary contribution mechanism and any Pareto-efficient mechanism. Given $\alpha \in]0,1[$ and $n \geq 2$, let $y^{\mathrm{PE}}(\alpha,n)$ (respectively, $y^{\mathrm{V}}(\alpha,n)$) be the equilibrium expected provision level of a public good under any Pareto-efficient mechanism (respectively, the voluntary contribution mechanism). To compare $y^{\mathrm{PE}}(\alpha,n)$ and $y^{\mathrm{V}}(\alpha,n)$, we

	α								
n	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2	1.0000	1.0000	1.0000	0.9861	0.9252	0.9373	0.9506	0.9653	0.9817
3	1.0000	1.0000	1.0822	1.0547	1.0435	1.0388	1.0374	1.0370	1.0344
4	1.0000	1.2778	1.1836	1.1424	1.1217	1.1107	1.1044	1.0996	1.0909
5	1.1612	1.3461	1.2439	1.1958	1.1706	1.1567	1.1489	1.1431	1.1332
6	1.4439	1.3915	1.2836	1.2313	1.2035	1.1883	1.1799	1.1742	1.1646
7	1.6771	1.4235	1.3117	1.2566	1.2271	1.2111	1.2026	1.1974	1.1886
8	1.7065	1.4473	1.3325	1.2755	1.2449	1.2284	1.2199	1.2152	1.2074
9	1.7292	1.4657	1.3487	1.2901	1.2587	1.2419	1.2336	1.2293	1.2224
10	1.7472	1.4804	1.3615	1.3018	1.2698	1.2528	1.2446	1.2408	1.2348
20	1.8267	1.5452	1.4186	1.3541	1.3196	1.3020	1.2949	1.2941	1.2936
30	1.8527	1.5665	1.4373	1.3713	1.3362	1.3185	1.3120	1.3125	1.3143
40	1.8656	1.5770	1.4467	1.3799	1.3444	1.3268	1.3206	1.3218	1.3249
50	1.8733	1.5833	1.4522	1.3851	1.3494	1.3318	1.3258	1.3274	1.3313
100	1.8886	1.5958	1.4633	1.3953	1.3593	1.3417	1.3362	1.3387	1.3442
500	1.8999	1.6031	1.4728	1.4065	1.3675	1.3517	1.3467	1.3464	1.3547

Table 3. Numerical results: the participation probability ratio, $PP(\alpha, n)$.

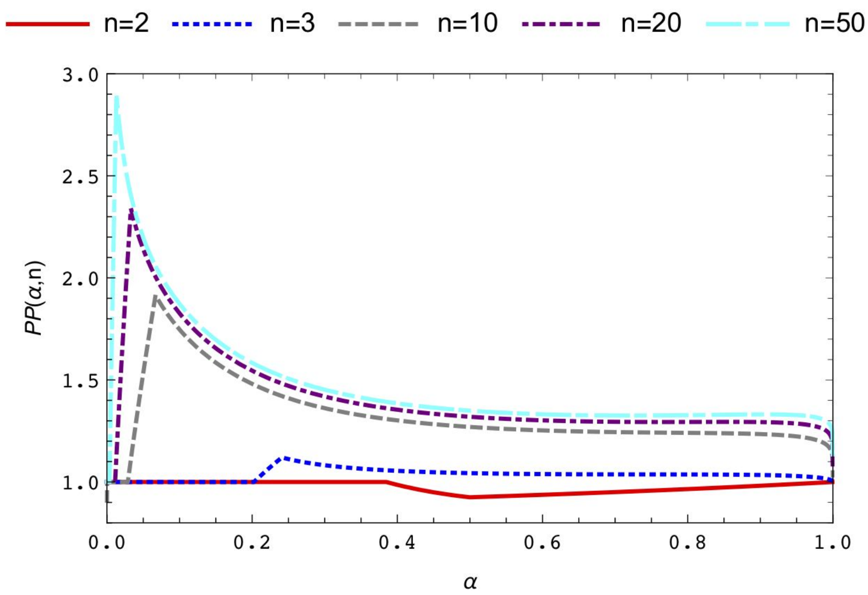

Figure 8. Graphs of $PP(\alpha, n)$ for $n = 2, 3, 10, 20, 50$.

introduce the following notion. Given $\alpha \in \,]0,1[$ and $n \geq 2$, let

$$PL(\alpha, n) \equiv \frac{y^{\mathrm{V}}(\alpha, n)}{y^{\mathrm{PE}}(\alpha, n)}$$

be the ratio of the equilibrium expected provision level of a public good under the voluntary contribution mechanism to that under any Pareto-efficient mechanism for (α, n). We call this the *expected provision level ratio.*

Table 4 reports the expected provision level ratio $PL(\alpha, n)$ when α varies from 0.1 to 0.9 and n from 2 to 500. Figure 9 shows the graphs of $PL(\alpha, n)$ when $n = 2$, 3, 10, 20, 50 and α varies from 0 to 1. Then, both Table 4 and Figure 9 reveal that, if $\alpha = 0.1$ and $n \geq 7$, then $PL(\alpha, n) > 1$ (in Table 4, $PL(\alpha, n) > 1$ is highlighted in gray), that is, the equilibrium expected provision level of a public good under the voluntary contribution mechanism is higher than that under any Pareto-efficient mechanism.

5.3 Equilibrium expected payoff

Finally, we numerically compute the equilibrium expected payoffs under the voluntary contribution mechanism and any Pareto-efficient mechanism. Given $\alpha \in \,]0,1[$ and $n \geq 2$, let $U^{\mathrm{PE}}(\alpha, n)$ (respectively, $U^{\mathrm{V}}(\alpha, n)$) be the expected payoff under any Pareto-efficient mechanism (respectively, the voluntary contribution mechanism). To compare $U^{\mathrm{PE}}(\alpha, n)$ and $U^{\mathrm{V}}(\alpha, n)$, we introduce the following notion. Given $\alpha \in \,]0,1[$ and $n \geq 2$, let

$$EP(\alpha, n) \equiv \frac{U^{\mathrm{V}}(\alpha, n)}{U^{\mathrm{PE}}(\alpha, n)}$$

be the ratio of the equilibrium expected payoff under the voluntary contribution mechanism to that under any Pareto-efficient mechanism for (α, n). We call this the *expected payoff ratio.*

Table 5 reports the expected payoff ratio $EP(\alpha, n)$ when α varies from 0.1 to 0.9 and n from 2 to 500. Figure 10 shows the graphs of $EP(\alpha, n)$ when $n = 2$, 3, 10, 20, 50 and α varies from 0 to 1. Then, both Table 5 and Figure 10 reveal that, if either (i) $\alpha = 0.1$ and $n \geq 6$ or (ii) $\alpha = 0.2$ and $n \geq 20$, then $EP(\alpha, n) > 1$ (in Table 5, $EP(\alpha, n) > 1$ is highlighted in gray), that is, the equilibrium expected payoff under the voluntary contribution mechanism is higher than that under any Pareto-efficient mechanism.[15]

[15] Strictly speaking, in the case of $\alpha = 0.2$, the expected payoff ratio is greater than 1 when there are at least 11 agents ($EP(0.2, 11) \approx 1.0010 > 1$).

	α								
n	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2	0.9091	0.8333	0.7692	0.7083	0.6399	0.6203	0.5971	0.5697	0.4886
3	0.8333	0.7143	0.7242	0.6843	0.6474	0.6081	0.5637	0.5116	0.4478
4	0.7692	0.8369	0.7637	0.7123	0.6656	0.6172	0.5631	0.5000	0.4216
5	0.8295	0.8672	0.7872	0.7298	0.6779	0.6246	0.5653	0.4960	0.4098
6	0.9626	0.8870	0.8027	0.7415	0.6865	0.6300	0.5674	0.4944	0.4032
7	1.0603	0.9009	0.8135	0.7499	0.6926	0.6340	0.5693	0.4936	0.3990
8	1.0742	0.9112	0.8216	0.7561	0.6973	0.6372	0.5708	0.4932	0.3962
9	1.0848	0.9192	0.8278	0.7609	0.7009	0.6397	0.5720	0.4930	0.3941
10	1.0932	0.9254	0.8328	0.7648	0.7038	0.6417	0.5731	0.4929	0.3925
20	1.1300	0.9531	0.8546	0.7820	0.7170	0.6509	0.5781	0.4931	0.3863
30	1.1419	0.9620	0.8617	0.7876	0.7213	0.6540	0.5799	0.4933	0.3845
40	1.1478	0.9665	0.8652	0.7904	0.7235	0.6556	0.5808	0.4935	0.3836
50	1.1513	0.9691	0.8674	0.7921	0.7248	0.6566	0.5814	0.4936	0.3831
100	1.1583	0.9744	0.8715	0.7955	0.7275	0.6585	0.5825	0.4938	0.3822
500	1.1638	0.9786	0.8749	0.7981	0.7295	0.6600	0.5835	0.4940	0.3814

Table 4. Numerical results: the expected provision level ratio, $PL(\alpha, n)$.

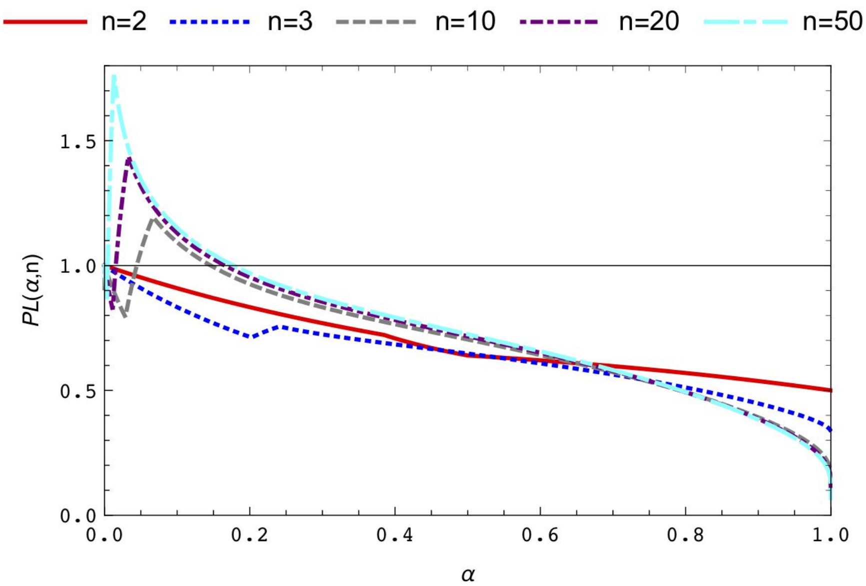

Figure 9. Graphs of $PL(\alpha, n)$ for $n = 2, 3, 10, 20, 50$.

n	α								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2	0.9734	0.9572	0.9470	0.9385	0.9251	0.9373	0.9506	0.9653	0.9817
3	0.9301	0.8898	0.9190	0.9139	0.9358	0.9248	0.9358	0.9505	0.9703
4	0.8836	0.9697	0.9309	0.9195	0.9187	0.9234	0.9324	0.9458	0.9658
5	0.9332	0.9801	0.9376	0.9232	0.9201	0.9232	0.9310	0.9436	0.9634
6	1.0426	0.9868	0.9419	0.9256	0.9212	0.9233	0.9302	0.9423	0.9619
7	1.1226	0.9914	0.9449	0.9273	0.9220	0.9234	0.9298	0.9414	0.9609
8	1.1278	0.9947	0.9471	0.9286	0.9227	0.9235	0.9295	0.9408	0.9602
9	1.1318	0.9973	0.9487	0.9296	0.9231	0.9236	0.9293	0.9404	0.9596
10	1.1349	0.9993	0.9500	0.9304	0.9235	0.9237	0.9291	0.9400	0.9592
20	1.1481	1.0080	0.9558	0.9338	0.9253	0.9242	0.9286	0.9386	0.9574
30	1.1524	1.0108	0.9576	0.9350	0.9260	0.9244	0.9285	0.9382	0.9568
40	1.1544	1.0122	0.9584	0.9356	0.9262	0.9246	0.9284	0.9380	0.9566
50	1.1556	1.0130	0.9590	0.9358	0.9264	0.9246	0.9283	0.9379	0.9564
100	1.1581	1.0146	0.9601	0.9364	0.9269	0.9469	0.9284	0.9376	0.9560
500	1.1596	1.0146	0.9613	0.9382	0.9269	0.9255	0.9288	0.9371	0.9559

Table 5. Numerical results: the expected payoff ratio, $EP(\alpha, n)$.

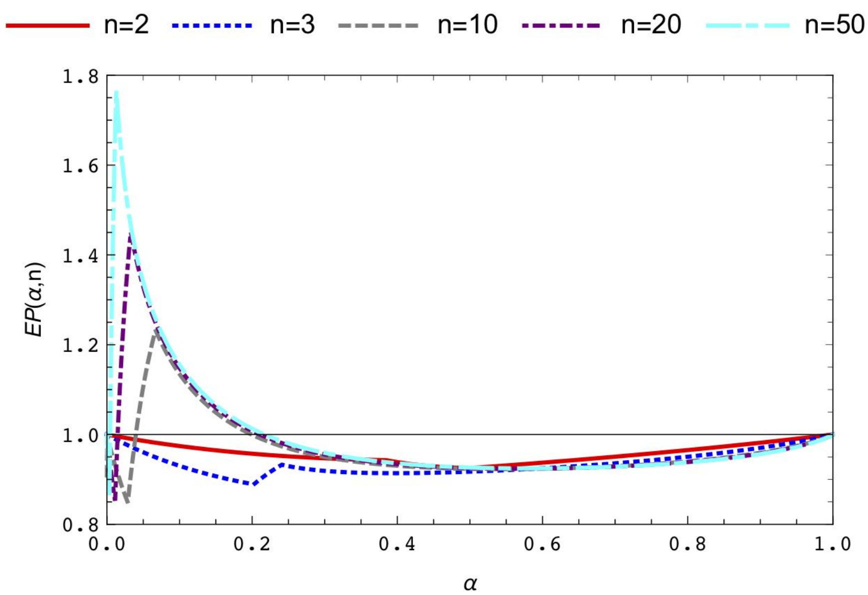

Figure 10. Graphs of $EP(\alpha, n)$ for $n = 2$, 3, 10, 20, 50.

6 Concluding remarks

We compared the performance of the voluntary contribution mechanism with that of any Pareto-efficient mechanism when each agent can choose whether she participates in a mechanism. In our two-stage game with voluntary participation, the equilibrium participation probability under the voluntary contribution mechanism becomes greater than that under any Pareto-efficient mechanism as the number of agents in the economy increases. Moreover, both the equilibrium expected payoff of each agent and the equilibrium expected provision level of the public good under the voluntary contribution mechanism become higher than those under any Pareto-efficient mechanism when the number of agents and the value of the public good are sufficiently large.

These results under voluntary participation contrast with those under compulsory participation: both the Nash equilibrium payoff of each agent and the Nash equilibrium provision level of the public good under the voluntary contribution mechanism are always lower than those under any Pareto-efficient mechanism when all agents are compelled to participate in the mechanisms. Our results suggest that the voluntary contribution mechanism, which cannot realize Pareto-efficient allocations under compulsory participation, might be superior to any Pareto-efficient mechanism when all agents have the ability not to participate. This leads us to re-examine the performances of mechanisms that are well behaved under compulsory participation.

However, there remain several open questions to be examined.

1. *Other classes of preferences.* We focused on symmetric Cobb-Douglas preferences but it would be interesting to examine what happens for other classes of preferences, such as quasi-linear and CES preferences.

2. *Public project problem.* We considered that the amount of the public good is continuous, whereas Palfrey and Rosenthal (1984), Dixit and Olson (2000), and Koriyama (2009) investigated participation games for a discrete public project. As such, we could compare the performance of the voluntary contribution (or provision point) mechanism with that of a Pareto-efficient mechanism in a public project environment with voluntary participation.

3. *Considering a more general class of mechanisms.* We focused on comparing two types of mechanisms, the voluntary contribution mechanism and any Pareto-efficient mechanism. Saijo and Yamato (2010) studied a wide class of mechanisms that are necessarily neither individually rational nor Pareto-efficient and established impossibility results on voluntary participation under the mechanisms in

the class. As such, we could examine which mechanism in a larger class is the second-best under voluntary participation.

4. *Political process.* Johansen (1977) argued that there is little evidence that public goods have never been provided through revelation of preferences. In fact, preference revelation mechanisms seem less practical because the mechanism designer has to spend a fair amount of time and effort to collect information from each agent. Johansen pointed out that in many realistic environments, public goods have been provided through political processes such as representative democracy. Thus, it is an interesting open question to incorporate political processes and examine a two-stage game on voluntary participation in a mechanism for providing a non-excludable public good.

5. *Experimental studies.* It would be intriguing to conduct an experimental comparison of the voluntary contribution mechanism and a Pareto-efficient mechanism with voluntary participation to test the validity of our theoretical results and verify which mechanism would work better in a laboratory.

These questions are scope for future research.

A Appendix: Proof of Theorem 1

Let

$$\begin{aligned} h^{\mathrm{PE}}(p,\alpha,n) &\equiv \left[U^{\mathrm{PE}}_{\mathsf{P}}(p,\alpha,n) - U^{\mathrm{PE}}_{\mathsf{NP}}(p,\alpha,n)\right]\frac{1}{\omega(1-\alpha)^{1-\alpha}} \\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}p^k(1-p)^{n-1-k}g^{\mathrm{PE}}(\alpha,k), \end{aligned}$$

where

$$g^{\mathrm{PE}}(\alpha,k) \equiv \alpha^{\alpha}(k+1)^{1-\alpha} - k^{1-\alpha}.$$

Before proving Theorem 1, we provide a useful lemma.

Lemma 1. *For each* $\alpha \in \,]0,1[$, $\frac{\partial g^{\mathrm{PE}}}{\partial k}(\alpha,k) < 0$.

Proof. Differentiating g^{PE} with respect to k, we obtain

$$\frac{\partial g^{\mathrm{PE}}}{\partial k}(\alpha,k) = \alpha^{\alpha}(1-\alpha)(k+1)^{-\alpha} - (1-\alpha)k^{-\alpha}$$

$$= (1-\alpha)\left[\left(\frac{\alpha}{k+1}\right)^{\alpha} - \left(\frac{1}{k}\right)^{\alpha}\right].$$

Since $\alpha \in]0,1[$, we have $\left(\frac{\alpha}{k+1}\right)^{\alpha} < \left(\frac{1}{k}\right)^{\alpha}$, which implies $\frac{\partial g^{\mathrm{PE}}}{\partial k}(\alpha, k) < 0$. $\square$

A.1 Proof of statement (i)

Differentiation of h^{PE} with respect to p yields

$$\frac{\partial h^{\mathrm{PE}}}{\partial p}(p,\alpha,n) = \sum_{k=0}^{n-2}(n-1)\binom{n-2}{k}p^k(1-p)^{n-2-k}\left[g^{\mathrm{PE}}(\alpha,k+1) - g^{\mathrm{PE}}(\alpha,k)\right].$$

It then follows from Lemma 1 that $g^{\mathrm{PE}}(\alpha, k+1) - g^{\mathrm{PE}}(\alpha, k) < 0$. This implies $\frac{\partial h^{\mathrm{PE}}}{\partial p}(p,\alpha,n) < 0$. Note that $\lim_{p\downarrow 0} h^{\mathrm{PE}}(p,\alpha,n) = \alpha^{\alpha} > 0$. If there is $\hat{p} \in]0,1[$ with $h^{\mathrm{PE}}(\hat{p},\alpha,n) = 0$, then the intermediate-value theorem ensures that such $\hat{p}$ uniquely exists. If there is no $\hat{p} \in]0,1[$ with $h^{\mathrm{PE}}(\hat{p},\alpha,n) = 0$, then for each $p \in [0,1]$, $h^{\mathrm{PE}}(p,\alpha,n) > 0$, that is, $U_{\mathsf{P}}^{\mathrm{PE}}(p,\alpha,n) > U_{\mathsf{NP}}^{\mathrm{PE}}(p,\alpha,n)$. This implies $p^{\mathrm{PE}}(\alpha,n) = 1$. Hence, we can conclude that there is a unique symmetric mixed strategy equilibrium. $\square$

A.2 Proof of statement (ii)

Note that $\lim_{p\downarrow 0} h^{\mathrm{PE}}(p,\alpha,n) = \lim_{p\downarrow 0} h^{\mathrm{PE}}(p,\alpha,n+1) = \alpha^{\alpha} > 0$ and both $h^{\mathrm{PE}}(\,\cdot\,,\alpha,n)$ and $h^{\mathrm{PE}}(\,\cdot\,,\alpha,n+1)$ are continuous in p. Moreover, as we have shown in the proof of statement (i) of Theorem 1, $\frac{\partial h^{\mathrm{PE}}}{\partial p}(p,\alpha,n) < 0$ and $\frac{\partial h^{\mathrm{PE}}}{\partial p}(p,\alpha,n+1) < 0$. Thus, in order to show that $p^{\mathrm{PE}}(\alpha,n) > p^{\mathrm{PE}}(\alpha,n+1)$ whenever $p^{\mathrm{PE}}(\alpha,n+1) < 1$, it is sufficient to show that $h^{\mathrm{PE}}(p,\alpha,n+1) = 0$ implies $h^{\mathrm{PE}}(p,\alpha,n) > 0$.

Suppose that $h^{\mathrm{PE}}(p,\alpha,n+1) = 0$. Let $\Delta^{\mathrm{PE}}(p,\alpha,n) \equiv 2 \times (1-p)h^{\mathrm{PE}}(p,\alpha,n) - h^{\mathrm{PE}}(p,\alpha,n+1)$. Since $h^{\mathrm{PE}}(p,\alpha,n+1) = 0$, $\Delta^{\mathrm{PE}}(p,\alpha,n) > 0$ implies $h^{\mathrm{PE}}(p,\alpha,n) > 0$. Thus, we now show that $\Delta^{\mathrm{PE}}(p,\alpha,n) > 0$. Then, $\Delta^{\mathrm{PE}}(p,\alpha,n)$ can be rewritten as

$$\begin{aligned}
\Delta^{\mathrm{PE}}(p,\alpha,n) &= 2(1-p)h^{\mathrm{PE}}(p,\alpha,n) - \sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}g^{\mathrm{PE}}(\alpha,k) \\
&= 2(1-p)h^{\mathrm{PE}}(p,\alpha,n) - p^n g^{\mathrm{PE}}(\alpha,n) - \sum_{k=0}^{n-1}\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}g^{\mathrm{PE}}(\alpha,k) \\
&= 2(1-p)h^{\mathrm{PE}}(p,\alpha,n) - p^n g^{\mathrm{PE}}(\alpha,n)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{k=0}^{n-1} \left[\frac{(n-1)!}{(k-1)!(n-k-1)!} \left(\frac{1}{k} + \frac{1}{n-k} \right) \right] p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k) \\
&= 2(1-p) h^{\mathrm{PE}}(p, \alpha, n) - p^n g^{\mathrm{PE}}(\alpha, n) \\
&- \sum_{k=0}^{n-1} \left[\frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \right] p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k) \\
&= (1-p) h^{\mathrm{PE}}(p, \alpha, n) + (1-p) \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} g^{\mathrm{PE}}(\alpha, k) \\
&- p^n g^{\mathrm{PE}}(\alpha, n) - \sum_{k=0}^{n-1} \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k) \\
&= (1-p) h^{\mathrm{PE}}(p, \alpha, n) + \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k) \\
&- p^n g^{\mathrm{PE}}(\alpha, n) - \sum_{k=0}^{n-1} \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k) \\
&= (1-p) h^{\mathrm{PE}}(p, \alpha, n) - p^n g^{\mathrm{PE}}(\alpha, n) \\
&- \sum_{k=0}^{n-1} \left[\binom{n-1}{k} + \binom{n-1}{k-1} - \binom{n-1}{k} \right] p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k) \\
&= (1-p) \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} g^{\mathrm{PE}}(\alpha, k) - p^n g^{\mathrm{PE}}(\alpha, n) \\
&- \sum_{k=0}^{n-1} \binom{n-1}{k-1} p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k) \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k) - p^n g^{\mathrm{PE}}(\alpha, n) \\
&- \sum_{k=0}^{n-1} \binom{n-1}{k-1} p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k) \\
&= \sum_{k=0}^{n-1} \left[\binom{n-1}{k} - \binom{n-1}{k-1} \right] p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k) - p^n g^{\mathrm{PE}}(\alpha, n) \\
&= \sum_{k=0}^{n-1} \left[\frac{(n-1)!}{k!(n-1-k)!} - \frac{(n-1)!}{(k-1)!(n-k)!} \right] p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k) - p^n g^{\mathrm{PE}}(\alpha, n) \\
&= \sum_{k=0}^{n-1} \left[\frac{(n-1)!(n-k)}{k!(n-k)!} - \frac{(n-1)!k}{k!(n-k)!} \right] p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k) - p^n g^{\mathrm{PE}}(\alpha, n) \\
&= \sum_{k=0}^{n-1} \left[\frac{(n-1)!(n-2k)}{k!(n-k)!} \right] p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k) - p^n g^{\mathrm{PE}}(\alpha, n)
\end{aligned}
$$

$$= \sum_{k=0}^{n-1} \left[\frac{n-2k}{n} \frac{n!}{k!(n-k)!} \right] p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k) - p^n g^{\mathrm{PE}}(\alpha, n)$$
$$= \sum_{k=0}^{n-1} \frac{n-2k}{n} \binom{n}{k} p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k) - p^n g^{\mathrm{PE}}(\alpha, n)$$
$$= \sum_{k=0}^{n} \frac{n-2k}{n} \binom{n}{k} p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k).$$

Let $k^* > 0$ be such that $g^{\mathrm{PE}}(\alpha, k^*) < 0$ and $g^{\mathrm{PE}}(\alpha, k^* - 1) \geq 0$. Then, Lemma 1, together with $h^{\mathrm{PE}}(p, \alpha, n+1) = 0$, implies that such k^* exists and

G1. for each $k \in \{0, \ldots, k^* - 2\}$, $g^{\mathrm{PE}}(\alpha, k) > 0$;

G2. $g^{\mathrm{PE}}(\alpha, k^* - 1) \geq 0$; and

G3. for each $k \in \{k^*, \ldots, n\}$, $g^{\mathrm{PE}}(\alpha, k) < 0$.

Let $\lambda \equiv \frac{n-2k^*}{n}$. For each $x \in \mathbb{R}$, let $\lceil x \rceil$ be the smallest integer greater than or equal to x. For each $\{r, t\} \subset \mathbb{R}$ with $0 \leq r \leq t \leq n$, let

$$\Delta_{r,t}^{\mathrm{PE}}(p, \alpha, n) \equiv \sum_{k=r}^{t} \frac{n-2k}{n} \binom{n}{k} p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k); \tag{5}$$
$$h_{r,t}^{\mathrm{PE}}(p, \alpha, n+1) \equiv \sum_{k=r}^{t} \binom{n}{k} p^k (1-p)^{n-k} g^{\mathrm{PE}}(\alpha, k). \tag{6}$$

There are three cases.

- **Case 1: $k^* < \frac{n}{2}$.** By using (6), $h^{\mathrm{PE}}(p, \alpha, n+1)$ can be rewritten as

$$h^{\mathrm{PE}}(p, \alpha, n+1) = \underbrace{h_{0,k^*-1}^{\mathrm{PE}}(p, \alpha, n+1)}_{\geq 0 \text{ by G1 and G2}} + \underbrace{h_{k^*, \lceil \frac{n}{2} \rceil - 1}^{\mathrm{PE}}(p, \alpha, n+1)}_{< 0 \text{ by G3}} + \underbrace{h_{\lceil \frac{n}{2} \rceil, n}^{\mathrm{PE}}(p, \alpha, n+1)}_{< 0 \text{ by G3}}.$$

Since $h^{\mathrm{PE}}(p, \alpha, n+1) = 0$,

$$h_{0,k^*-1}^{\mathrm{PE}}(p, \alpha, n+1) = -\left[h_{k^*, \lceil \frac{n}{2} \rceil - 1}^{\mathrm{PE}}(p, \alpha, n+1) + h_{\lceil \frac{n}{2} \rceil, n}^{\mathrm{PE}}(p, \alpha, n+1) \right] > 0.$$

Note that $\lambda = \frac{n-2k^*}{n} > 0$ because $n > 2k^*$. Then,

$$\lambda h_{0,k^*-1}^{\mathrm{PE}}(p, \alpha, n+1) = -\lambda \left[h_{k^*, \lceil \frac{n}{2} \rceil - 1}^{\mathrm{PE}}(p, \alpha, n+1) + h_{\lceil \frac{n}{2} \rceil, n}^{\mathrm{PE}}(p, \alpha, n+1) \right]. \tag{7}$$

Since $-\lambda h^{\mathrm{PE}}_{\lceil \frac{n}{2} \rceil, n}(p,\alpha,n+1) > 0$, (7) implies that

$$\lambda h^{\mathrm{PE}}_{0,k^*-1}(p,\alpha,n+1) > -\lambda h^{\mathrm{PE}}_{k^*,\lceil \frac{n}{2} \rceil -1}(p,\alpha,n+1). \tag{8}$$

Note that by $n > 2k^*$,

- for each $k \in \{0,\ldots,k^*-1\}$, $\frac{n-2k}{n} > \frac{n-2k^*}{n} = \lambda$;
- for each $k \in \left\{k^*,\ldots,\lceil \frac{n}{2} \rceil - 1\right\}$, $\lambda = \frac{n-2k^*}{n} \geq \frac{n-2k}{n}$.

It then follows from (5) and (8) that

$$\begin{aligned}
\Delta^{\mathrm{PE}}_{0,k^*-1}(p,\alpha,n) &> \lambda h^{\mathrm{PE}}_{0,k^*-1}(p,\alpha,n+1) \\
&> -\lambda h^{\mathrm{PE}}_{k^*,\lceil \frac{n}{2} \rceil -1}(p,\alpha,n+1) \\
&\geq -\Delta^{\mathrm{PE}}_{k^*,\lceil \frac{n}{2} \rceil -1}(p,\alpha,n).
\end{aligned}$$

Hence, $\Delta^{\mathrm{PE}}(p,\alpha,n) = \underbrace{\Delta^{\mathrm{PE}}_{0,k^*-1}(p,\alpha,n) + \Delta^{\mathrm{PE}}_{k^*,\lceil \frac{n}{2} \rceil -1}(p,\alpha,n)}_{>0} + \underbrace{\Delta^{\mathrm{PE}}_{\lceil \frac{n}{2} \rceil, n}(p,\alpha,n)}_{>0} > 0.$

- **Case 2: $\boldsymbol{k^* > \frac{n}{2}}$.** By using (6), $h^{\mathrm{PE}}(p,\alpha,n+1)$ can be rewritten as

$$h^{\mathrm{PE}}(p,\alpha,n+1) = \underbrace{h^{\mathrm{PE}}_{0,\lceil \frac{n}{2} \rceil -2}(p,\alpha,n+1)}_{>0 \text{ by G1}} + \underbrace{h^{\mathrm{PE}}_{\lceil \frac{n}{2} \rceil -1,k^*-1}(p,\alpha,n+1)}_{\geq 0 \text{ by G1 and G2}} + \underbrace{h^{\mathrm{PE}}_{k^*,n}(p,\alpha,n+1)}_{<0 \text{ by G3}}.$$

Since $h^{\mathrm{PE}}(p,\alpha,n+1) = 0$,

$$-\left[h^{\mathrm{PE}}_{0,\lceil \frac{n}{2} \rceil -2}(p,\alpha,n+1) + h^{\mathrm{PE}}_{\lceil \frac{n}{2} \rceil -1,k^*-1}(p,\alpha,n+1)\right] = h^{\mathrm{PE}}_{k^*,n}(p,\alpha,n+1).$$

Note that $\lambda = \frac{n-2k^*}{n} < 0$ because $2k^* > n$. It then follows that

$$-\lambda\left[h^{\mathrm{PE}}_{0,\lceil \frac{n}{2} \rceil -2}(p,\alpha,n+1) + h^{\mathrm{PE}}_{\lceil \frac{n}{2} \rceil -1,k^*-1}(p,\alpha,n+1)\right] = \lambda h^{\mathrm{PE}}_{k^*,n}(p,\alpha,n+1) > 0. \tag{9}$$

Since $-\lambda h^{\mathrm{PE}}_{0,\lceil \frac{n}{2} \rceil -2}(p,\alpha,n+1) > 0$, (9) implies that

$$-\lambda h^{\mathrm{PE}}_{\lceil \frac{n}{2} \rceil -1,k^*-1}(p,\alpha,n+1) < \lambda h^{\mathrm{PE}}_{k^*,n}(p,\alpha,n+1). \tag{10}$$

Note that by $2k^* > n$,

- for each $k \in \{k^*+1,\ldots,n\}$, $0 > \lambda = \frac{n-2k^*}{n} > \frac{n-2k}{n}$;

- for each $k \in \left\{\left\lceil \frac{n}{2} \right\rceil - 1, \ldots, k^* - 1\right\}$, $\frac{n-2k}{n} > \frac{n-2k^*}{n} = \lambda$.

It then follows from (5) and (10) that

$$\begin{aligned}\Delta^{\mathrm{PE}}_{k^*,n}(p,\alpha,n) &> \lambda h^{\mathrm{PE}}_{k^*,n}(p,\alpha,n+1) \\ &> -\lambda h^{\mathrm{PE}}_{\lceil \frac{n}{2} \rceil -1,k^*-1}(p,\alpha,n+1) \\ &\geq -\Delta^{\mathrm{PE}}_{\lceil \frac{n}{2} \rceil -1,k^*-1}(p,\alpha,n).\end{aligned}$$

Hence, $\Delta^{\mathrm{PE}}(p,\alpha,n) = \underbrace{\Delta^{\mathrm{PE}}_{0,\lceil \frac{n}{2} \rceil -2}(p,\alpha,n)}_{>0} + \underbrace{\Delta^{\mathrm{PE}}_{\lceil \frac{n}{2} \rceil -1,k^*-1}(p,\alpha,n) + \Delta^{\mathrm{PE}}_{k^*,n}(p,\alpha,n)}_{>0} > 0$.

• **Case 3: $\boldsymbol{k^* = \frac{n}{2}}$.** Then, $\Delta^{\mathrm{PE}}_{k^*,k^*} = \frac{n-2k^*}{n}\binom{n-1}{k^*}p^{k^*}(1-p)^{n-k^*}g^{\mathrm{PE}}(\alpha,k^*) = 0$. Thus, we obtain $\Delta^{\mathrm{PE}}(p,\alpha,n) = \Delta^{\mathrm{PE}}_{0,k^*-1}(p,\alpha,n) + \Delta^{\mathrm{PE}}_{k^*+1,n}(p,\alpha,n)$. Note that by $2k^* = n$,

- for each $k \in \{0, \ldots, k^* - 1\}$, $\frac{n-2k}{n} > 0$;
- for each $k \in \{k^* + 1, \ldots, n\}$, $\frac{n-2k}{n} < 0$.

Hence, by (5) and G1–G3, $\Delta^{\mathrm{PE}}(p,\alpha,n) = \underbrace{\Delta^{\mathrm{PE}}_{0,k^*-1}(p,\alpha,n)}_{>0} + \underbrace{\Delta^{\mathrm{PE}}_{k^*+1,n}(p,\alpha,n)}_{>0} > 0$.

From Cases 1–3, $\Delta^{\mathrm{PE}}(p,\alpha,n) > 0$. Recall that $\Delta^{\mathrm{PE}}(p,\alpha,n) = 2(1-p)h^{\mathrm{PE}}(p,\alpha,n) - h^{\mathrm{PE}}(p,\alpha,n+1)$. Since $h^{\mathrm{PE}}(p,\alpha,n+1) = 0$, $\Delta^{\mathrm{PE}}(p,\alpha,n) > 0$ implies $h^{\mathrm{PE}}(p,\alpha,n) > 0$.

Finally, we observe that $\lim_{k\to\infty} p^{\mathrm{PE}}(\alpha,k) = 0$ because the sequence $\{p^{\mathrm{PE}}(\alpha,k)\}_{k\geq 2}$ is monotonically decreasing and bounded by 0 from below. □

B Appendix: Proof of Theorem 2

Let

$$\begin{aligned}h^{\mathrm{V}}(p,\alpha,n) &\equiv \left[U^{\mathrm{V}}_{\mathsf{P}}(p,\alpha,n) - U^{\mathrm{V}}_{\mathsf{NP}}(p,\alpha,n)\right]\frac{1}{\omega(1-\alpha)^{1-\alpha}} \\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}p^k(1-p)^{n-1-k}g^{\mathrm{V}}(\alpha,k),\end{aligned}$$

where

$$g^{\mathrm{V}}(\alpha,k) \equiv \frac{\alpha^\alpha(k+1)}{1+k\alpha} - \left[\frac{k}{1+\alpha(k-1)}\right]^{1-\alpha}.$$

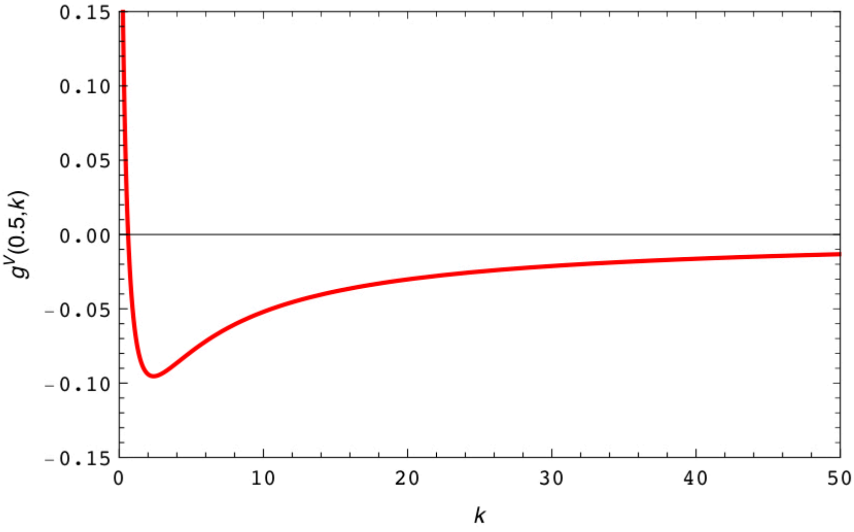

Figure 11. Graph of $g^{\mathrm{V}}(0.5, k)$.

Note that $\lim_{\alpha \downarrow 0} g^{\mathrm{V}}(\alpha, k) = 1$.

In contrast to the function g^{PE}, the function g^{V} is not monotonically decreasing in k and is a somewhat complicated form (Figure 11 illustrates the form of the function $g^{\mathrm{V}}(0.5, k)$). Due to this fact, we cannot prove Theorem 2 by applying the same proof techniques in Theorem 1. Thus, we provide three useful lemmas regarding the form of g^{V}. The first lemma (Lemma 2) states that for each $\alpha \in \,]0, 1[$, $g^{\mathrm{V}}(\alpha, 0)$ is positive. The second lemma (Lemma 3) states that for each $k \geq 1$, the graph of the function $g^{\mathrm{V}}(\,\cdot\,, k)$ intersects the horizontal axis only once at some value $\hat{\alpha} \in \,]0, 1[$. This, together with the facts that $g^{\mathrm{V}}(\,\cdot\,, k)$ is continuous in α and $\lim_{\alpha \downarrow 0} g^{\mathrm{V}}(\alpha, k) = 1$, implies that the derivative of $g^{\mathrm{V}}(\,\cdot\,, k)$ evaluated at $\hat{\alpha}$ is negative. The third lemma (Lemma 4) states that for each $\alpha \in \,]0, 1[$ and each $k \geq 0$, $g^{\mathrm{V}}(\alpha, k+1) = 0$ implies $g^{\mathrm{V}}(\alpha, k) > 0$. Figure 12 illustrates these lemmas. By these three lemmas, for each $\alpha \in \,]0, 1[$, if $h^{\mathrm{V}}(p, \alpha, n) = 0$, then we can find $k^* > 0$ such that if $k \geq k^*$, $g^{\mathrm{V}}(\alpha, k) < 0$; otherwise, $g^{\mathrm{V}}(\alpha, k) \geq 0$. By invoking this fact, we can prove Theorem 2 in a similar way in the proof of Theorem 1.

Lemma 2. *For each $\alpha \in \,]0, 1[$, $g^{\mathrm{V}}(\alpha, 0) > 0$.*

Proof. Let $\alpha \in \,]0, 1[$. Then, $g^{\mathrm{V}}(\alpha, 0) = \alpha^{\alpha} > 0$. □

Lemma 3. *For each $k \geq 1$, there is a unique value $\hat{\alpha} \in \,]0, 1[$ such that $g^{\mathrm{V}}(\hat{\alpha}, k) = 0$. Moreover, $\frac{\partial g^{\mathrm{V}}}{\partial \alpha}(\hat{\alpha}, k) < 0$.*

Proof. Let

$$g^{\mathrm{V}}_{\oplus}(\alpha, k) \equiv \frac{\alpha^{\alpha}(k+1)}{1 + k\alpha} \quad \text{and} \quad g^{\mathrm{V}}_{\ominus}(\alpha, k) \equiv \left[\frac{k}{1 + (k-1)\alpha}\right]^{1-\alpha}.$$

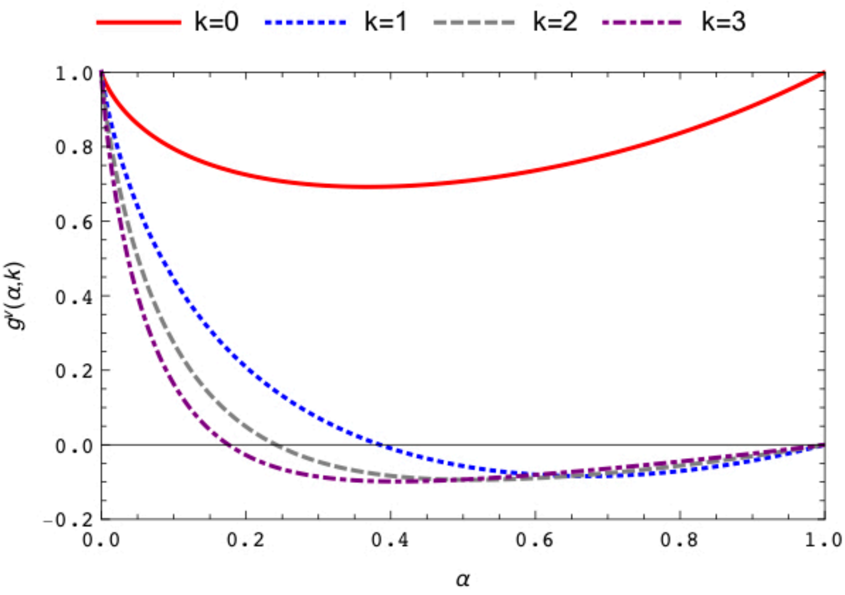

Figure 12. Graphs of $g^{\mathrm{V}}(\alpha,k)$ for $k=0,\,1,\,2,\,3$.

Note that $g^{\mathrm{V}}(\alpha,k)=g^{\mathrm{V}}_{\oplus}(\alpha,k)-g^{\mathrm{V}}_{\ominus}(\alpha,k)$. Then, we obtain

$$\begin{aligned}
\frac{\partial g^{\mathrm{V}}_{\oplus}}{\partial\alpha}(\alpha,k) &= -\frac{k(k+1)\alpha^{\alpha}}{(1+k\alpha)^2}+\frac{(k+1)\alpha^{\alpha}(1+\ln\alpha)}{1+k\alpha};\\
\frac{\partial^2 g^{\mathrm{V}}_{\oplus}}{\partial\alpha^2}(\alpha,k) &= \frac{2k^2\alpha^{\alpha}(k+1)}{(1+k\alpha)^3}+\frac{(k+1)\alpha^{\alpha-1}}{1+k\alpha}-\frac{2k(k+1)\alpha^{\alpha}(1+\ln\alpha)}{(1+k\alpha)^2}+\frac{(k+1)\alpha^{\alpha}(1+\ln\alpha)^2}{1+k\alpha}\\
&= \frac{(k+1)\alpha^{\alpha}}{1+k\alpha}\left\{\left[\frac{k}{1+k\alpha}-(\ln\alpha+1)\right]^2+\frac{k^2}{(1+k\alpha)^2}+\frac{1}{\alpha}\right\};\\
\frac{\partial g^{\mathrm{V}}_{\ominus}}{\partial\alpha}(\alpha,k) &= -\left[\frac{k}{1+(k-1)\alpha}\right]^{1-\alpha}\left[\ln\frac{k}{1+(k-1)\alpha}+\frac{(k-1)(\alpha-1)}{1+(k-1)\alpha}\right];\\
\frac{\partial^2 g^{\mathrm{V}}_{\ominus}}{\partial\alpha^2}(\alpha,k) &= \left\{\frac{2(k-1)}{1+(k-1)\alpha}+\frac{(1-\alpha)(k-1)^2}{[1+(k-1)\alpha]^2}\right\}\left[\frac{k}{1+(k-1)\alpha}\right]^{1-\alpha}\\
&\quad+\left[-\ln\frac{k}{1+(k-1)\alpha}-\frac{(k-1)(\alpha-1)}{1+(k-1)\alpha}\right]^2\left[\frac{k}{1+(k-1)\alpha}\right]^{1-\alpha}.
\end{aligned}$$

There are two cases.

- **Case 1: $\boldsymbol{k=1}$.** Then, $\lim_{\alpha\downarrow 0}\frac{\partial g^{\mathrm{V}}}{\partial\alpha}(\alpha,k)=-\infty<0$, $\lim_{\alpha\uparrow 1}\frac{\partial g^{\mathrm{V}}}{\partial\alpha}(\alpha,k)=\frac{1}{2}>0$, and $\frac{\partial^2 g^{\mathrm{V}}}{\partial\alpha^2}(\alpha,k)>0$. Since $\lim_{\alpha\downarrow 0} g^{\mathrm{V}}(\alpha,k)=1>0$, $\lim_{\alpha\uparrow 1} g^{\mathrm{V}}(\alpha,k)=0$, and g^{V} is continuous in α, these imply that there is a unique value $\hat{\alpha}\in\,]0,1[$ with $g^{\mathrm{V}}(\hat{\alpha},k)=0$.

- **Case 2: $\boldsymbol{k\geq 2}$.** Then, $\lim_{\alpha\downarrow 0}\frac{\partial g^{\mathrm{V}}_{\oplus}}{\partial\alpha}(\alpha,k)=-\infty<0$, $\lim_{\alpha\uparrow 1}\frac{\partial g^{\mathrm{V}}_{\oplus}}{\partial\alpha}(\alpha,k)=1-\frac{k}{1+k}>0$, $\lim_{\alpha\downarrow 0}\frac{\partial g^{\mathrm{V}}_{\ominus}}{\partial\alpha}(\alpha,k)=-k(k-1+\ln k)<0$, $\lim_{\alpha\uparrow 1}\frac{\partial g^{\mathrm{V}}_{\ominus}}{\partial\alpha}(\alpha,k)=0$, $\frac{\partial^2 g^{\mathrm{V}}_{\oplus}}{\partial\alpha^2}(\alpha,k)>0$, and

$\frac{\partial^2 g_{\ominus}^{\mathrm{V}}}{\partial \alpha^2}(\alpha, k) > 0$. These facts imply that there is a unique value $\hat{\alpha} \in]0,1[$ such that $g_{\oplus}^{\mathrm{V}}(\hat{\alpha}, k) = g_{\ominus}^{\mathrm{V}}(\hat{\alpha}, k)$, that is, $g^{\mathrm{V}}(\hat{\alpha}, k) = 0$, since $\lim_{\alpha \uparrow 1} g_{\oplus}^{\mathrm{V}}(\alpha, k) = \lim_{\alpha \uparrow 1} g_{\ominus}^{\mathrm{V}}(\alpha, k) = 1$, $\lim_{\alpha \downarrow 0} g_{\oplus}^{\mathrm{V}}(\alpha, k) = k + 1 > \lim_{\alpha \downarrow 0} g_{\ominus}^{\mathrm{V}}(\alpha, k) = k > 1$, and both $g_{\oplus}$ and $g_{\ominus}$ are continuous in α.

Moreover, $g^{\mathrm{V}}(\hat{\alpha}, k) = 0$ implies $\frac{\partial g^{\mathrm{V}}}{\partial \alpha}(\hat{\alpha}, k) < 0$ because $\lim_{\alpha \downarrow 0} g^{\mathrm{V}}(\alpha, k) > 0$ and g^{V} is continuous in α. $\square$

Lemma 4. *For each $\alpha \in]0,1[$ and each $k \geq 0$, if $g^{\mathrm{V}}(\alpha, k+1) = 0$, then $g^{\mathrm{V}}(\alpha, k) > 0$.*

Proof. Let $\alpha \in]0,1[$ and $k \geq 0$ be such that $g^{\mathrm{V}}(\alpha, k+1) = 0$. Then,

$$g^{\mathrm{V}}(\alpha, k+1) = \frac{\alpha^{\alpha}(k+2)}{1 + (k+1)\alpha} - \left(\frac{k+1}{1+k\alpha}\right)^{1-\alpha} = 0,$$

that is,

$$\alpha^{\alpha}(k+1) = \left(\frac{k+1}{1+k\alpha}\right)^{1-\alpha} [1 + (k+1)\alpha] - \alpha^{\alpha}. \tag{11}$$

By (11), $g^{\mathrm{V}}(\alpha, k)$ can be rewritten as

$$\begin{aligned}
g^{\mathrm{V}}(\alpha, k) &= \frac{\alpha^{\alpha}(k+1)}{1+k\alpha} - \left[\frac{k}{1+(k-1)\alpha}\right]^{1-\alpha} \\
&= \frac{1}{1+k\alpha}\left\{\left(\frac{k+1}{1+k\alpha}\right)^{1-\alpha}[1+(k+1)\alpha] - \alpha^{\alpha}\right\} - \left[\frac{k}{1+(k-1)\alpha}\right]^{1-\alpha} \\
&= \frac{(1+k\alpha)+\alpha}{1+k\alpha}\left(\frac{k+1}{1+k\alpha}\right)^{1-\alpha} - \frac{\alpha^{\alpha}}{1+k\alpha} - \left[\frac{k}{1+(k-1)\alpha}\right]^{1-\alpha} \\
&= \left(\frac{k+1}{1+k\alpha}\right)^{1-\alpha} + \frac{\alpha}{1+k\alpha}\left(\frac{k+1}{1+k\alpha}\right)^{1-\alpha} - \frac{\alpha^{\alpha}}{1+k\alpha} - \left[\frac{k}{1+(k-1)\alpha}\right]^{1-\alpha} \\
&= \left(\frac{k+1}{1+k\alpha}\right)^{1-\alpha} - \left[\frac{k}{1+(k-1)\alpha}\right]^{1-\alpha} \\
&\quad - \frac{\alpha}{1+k\alpha}\left[\left(\frac{1}{\alpha}\right)^{1-\alpha} - \left(\frac{k+1}{1+k\alpha}\right)^{1-\alpha}\right].
\end{aligned} \tag{12}$$

Let $a \equiv \frac{1}{\alpha}$, $b \equiv \frac{k+1}{1+k\alpha}$, and $c \equiv \frac{k}{1+(k-1)\alpha}$. Note that $a > b > c$ and

$$\begin{aligned}
b - c &= \frac{k+1}{1+k\alpha} - \frac{k}{1+(k-1)\alpha} \\
&= \frac{1-\alpha}{(1+k\alpha)[1+(k-1)\alpha]}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha}{1+(k-1)\alpha}\left[\frac{1-\alpha+k\alpha-k\alpha}{\alpha(1+k\alpha)}\right] \\
&= \frac{\alpha}{1+(k-1)\alpha}\left(\frac{1}{\alpha}-\frac{k+1}{1+k\alpha}\right) \\
&= \frac{\alpha}{1+(k-1)\alpha}\,(a-b)\,.
\end{aligned}
\tag{13}
$$

Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a function defined by $f(x) = x^{1-\alpha}$. Then, by (12),

$$
g^{\mathrm{V}}(\alpha, k) = [f(b) - f(c)] - \frac{\alpha}{1+k\alpha}[f(a) - f(b)]. \tag{14}
$$

Since f is strictly concave,

$$
\frac{f(b)-f(c)}{b-c} > \frac{f(a)-f(b)}{a-b}.
$$

By (13), this can be rewritten as

$$
\frac{f(b)-f(c)}{\frac{\alpha}{1+(k-1)\alpha}\,(a-b)} > \frac{f(a)-f(b)}{a-b}.
$$

Since $a-b>0$ and $\frac{\alpha}{1+(k-1)\alpha} > \frac{\alpha}{1+k\alpha}$,

$$
f(b)-f(c) > \frac{\alpha}{1+(k-1)\alpha}[f(a)-f(b)] > \frac{\alpha}{1+k\alpha}[f(a)-f(b)].
$$

By (14), this implies $g^{\mathrm{V}}(\alpha, k) > 0$. □

B.1 Proof of statement (i)

Note that $\lim_{p\downarrow 0} h^{\mathrm{V}}(p,\alpha,n) > 0$ and h^{V} is continuous in p. If there is no $\hat{p} \in\,]0,1[$ with $h^{\mathrm{V}}(\hat{p},\alpha,n) = 0$, then for each $p \in [0,1]$, $h^{\mathrm{V}}(p,\alpha,n) > 0$, that is, $U^{\mathrm{V}}_{\mathsf{P}}(p,\alpha,n) > U^{\mathrm{V}}_{\mathsf{NP}}(p,\alpha,n)$. This implies $p^{\mathrm{V}}(\alpha,n) = 1$.

We now consider the case where there is $\hat{p} \in\,]0,1[$ with $h^{\mathrm{V}}(\hat{p},\alpha,n) = 0$. Pick any $\hat{p} \in\,]0,1[$ with $h^{\mathrm{V}}(\hat{p},\alpha,n) = 0$. Then, it is sufficient to show that $\frac{\partial h^{\mathrm{V}}}{\partial p}(\hat{p},\alpha,n) < 0$, because this, together with the facts that $\lim_{p\downarrow 0} h^{\mathrm{V}}(p,\alpha,n) > 0$ and h^{V} is continuous in p, implies that there is a unique symmetric mixed strategy equilibrium. We proceed in two steps.

Step 1: $\boldsymbol{h^{\mathrm{V}}(\hat{p},\alpha,n-1) > 0}$. Since $\lim_{\alpha\downarrow 0} g^{\mathrm{V}}(\alpha,k) > 0$ and g^{V} is continuous in α, by Lemmas 3 and 4, $g^{\mathrm{V}}(\alpha,k) < 0$ implies $g^{\mathrm{V}}(\alpha,k+1) < 0$. This, together with $h^{\mathrm{V}}(\hat{p},\alpha,n) = 0$ and Lemma 2, implies that there exists $k^* > 0$ such that

- for each $k \in \{0, \ldots, k^* - 1\}$, $g^{\mathrm{V}}(\alpha, k) \geq 0$;
- for each $k \in \{k^*, \ldots, n-1\}$, $g^{\mathrm{V}}(\alpha, k) < 0$.

There are two cases.

• **Case 1: $\boldsymbol{k^* \neq n-1}$.** Then, $h^{\mathrm{V}}(\hat{p}, \alpha, n) = 0$ is equivalent to

$$\sum_{k=0}^{k^*-1} \binom{n-1}{k} \hat{p}^k (1-\hat{p})^{n-1-k} g^{\mathrm{V}}(\alpha, k) = -\sum_{k=k^*}^{n-1} \binom{n-1}{k} \hat{p}^k (1-\hat{p})^{n-1-k} g^{\mathrm{V}}(\alpha, k) > 0.$$

This equality can be rearranged to give

$$\begin{aligned}
&(n-1)(1-\hat{p}) \sum_{k=0}^{k^*-1} \frac{1}{n-1-k} \binom{n-2}{k} \hat{p}^k (1-\hat{p})^{n-2-k} g^{\mathrm{V}}(\alpha, k) \\
&\quad = -(n-1)(1-\hat{p}) \sum_{k=k^*}^{n-2} \frac{1}{n-1-k} \binom{n-2}{k} \hat{p}^k (1-\hat{p})^{n-2-k} g^{\mathrm{V}}(\alpha, k) - \hat{p}^{n-1} g^{\mathrm{V}}(\alpha, n-1),
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\sum_{k=0}^{k^*-1} \frac{1}{n-1-k} \binom{n-2}{k} \hat{p}^k (1-\hat{p})^{n-2-k} g^{\mathrm{V}}(\alpha, k) \\
&\quad = -\sum_{k=k^*}^{n-2} \frac{1}{n-1-k} \binom{n-2}{k} \hat{p}^k (1-\hat{p})^{n-2-k} g^{\mathrm{V}}(\alpha, k) + \gamma,
\end{aligned}$$

where $\gamma \equiv -\frac{\hat{p}^{n-1} g^{\mathrm{V}}(\alpha, n-1)}{(n-1)(1-\hat{p})} > 0$. Note that if $j > k^* > \ell$, then $\frac{1}{n-1-j} > \frac{1}{n-1-k^*} > \frac{1}{n-1-\ell}$. Therefore,

$$\begin{aligned}
&\sum_{k=0}^{k^*-1} \frac{1}{n-1-k^*} \binom{n-2}{k} \hat{p}^k (1-\hat{p})^{n-2-k} g^{\mathrm{V}}(\alpha, k) \\
&\quad > -\sum_{k=k^*}^{n-2} \frac{1}{n-1-k^*} \binom{n-2}{k} \hat{p}^k (1-\hat{p})^{n-2-k} g^{\mathrm{V}}(\alpha, k) + \frac{\gamma}{n-1-k^*}.
\end{aligned}$$

That is,

$$\sum_{k=0}^{k^*-1} \binom{n-2}{k} \hat{p}^k (1-\hat{p})^{n-2-k} g^{\mathrm{V}}(\alpha, k) > -\sum_{k=k^*}^{n-2} \binom{n-2}{k} \hat{p}^k (1-\hat{p})^{n-2-k} g^{\mathrm{V}}(\alpha, k) + \gamma.$$

Hence, we obtain

$$\begin{aligned}h^{\mathrm{V}}(\hat{p},\alpha,n-1) &= \sum_{k=0}^{k^*-1}\binom{n-2}{k}\hat{p}^k(1-\hat{p})^{n-2-k}g^{\mathrm{V}}(\alpha,k)+\sum_{k=k^*}^{n-2}\binom{n-2}{k}\hat{p}^k(1-\hat{p})^{n-2-k}g^{\mathrm{V}}(\alpha,k)\\ &> \gamma\\ &> 0,\end{aligned}$$

which is the desired conclusion.

- **Case 2: $\boldsymbol{k^* = n-1}$.** Then, $h^{\mathrm{V}}(\hat{p},\alpha,n)=0$ is equivalent to

$$\sum_{k=0}^{n-2}\binom{n-1}{k}\hat{p}^k(1-\hat{p})^{n-1-k}g^{\mathrm{V}}(\alpha,k) = -\hat{p}^{n-1}g^{\mathrm{V}}(\alpha,n-1)>0.$$

This equality can be rearranged to give

$$(n-1)(1-\hat{p})\sum_{k=0}^{n-2}\frac{1}{n-1-k}\binom{n-2}{k}\hat{p}^k(1-\hat{p})^{n-2-k}g^{\mathrm{V}}(\alpha,k) = -\hat{p}^{n-1}g^{\mathrm{V}}(\alpha,n-1),$$

or equivalently,

$$\sum_{k=0}^{n-2}\frac{1}{n-1-k}\binom{n-2}{k}\hat{p}^k(1-\hat{p})^{n-2-k}g^{\mathrm{V}}(\alpha,k) = \gamma,$$

where $\gamma \equiv -\frac{\hat{p}^{n-1}g^{\mathrm{V}}(\alpha,n-1)}{(n-1)(1-\hat{p})} > 0$. Note that if $n-2>k$, then $1>\frac{1}{n-1-k}$. Therefore,

$$\sum_{k=0}^{n-2}\binom{n-2}{k}\hat{p}^k(1-\hat{p})^{n-2-k}g^{\mathrm{V}}(\alpha,k) > \gamma > 0.$$

Since $h^{\mathrm{V}}(\hat{p},\alpha,n-1)=\sum_{k=0}^{n-2}\binom{n-2}{k}\hat{p}^k(1-\hat{p})^{n-2-k}g^{\mathrm{V}}(\alpha,k)$, this implies $h^{\mathrm{V}}(\hat{p},\alpha,n-1) > 0$.

Step 2: $\boldsymbol{\frac{\partial h^{\mathrm{V}}}{\partial p}(\hat{p},\alpha,n)<0}$. Then, $h^{\mathrm{V}}(p,\alpha,n)$ can be rewritten as

$$\begin{aligned}h^{\mathrm{V}}(p,\alpha,n) &= \sum_{k=0}^{n-1}\binom{n-1}{k}p^k(1-p)^{n-1-k}g^{\mathrm{V}}(\alpha,k)\\ &= p^{n-2}\left[\binom{n-1}{n-1}pg^{\mathrm{V}}(\alpha,n-1)+\binom{n-1}{n-2}(1-p)g^{\mathrm{V}}(\alpha,n-2)\right]\end{aligned}$$

$$
\begin{aligned}
&+ (1-p)^{n-2}\left[\binom{n-1}{0}(1-p)g^{\mathrm{V}}(\alpha,0) + \binom{n-1}{1}pg^{\mathrm{V}}(\alpha,1)\right] \\
&+ \sum_{k=2}^{n-3}\binom{n-1}{k}p^k(1-p)^{n-1-k}g^{\mathrm{V}}(\alpha,k) \\
={}& p^{n-2}\left[pg^{\mathrm{V}}(\alpha,n-1) + (n-1)(1-p)g^{\mathrm{V}}(\alpha,n-2)\right] \\
&+ (1-p)^{n-2}\left[(1-p)g^{\mathrm{V}}(\alpha,0) + (n-1)pg^{\mathrm{V}}(\alpha,1)\right] \\
&+ \sum_{k=2}^{n-3}\binom{n-1}{k}p^k(1-p)^{n-1-k}g^{\mathrm{V}}(\alpha,k) \\
={}& p^{n-2}\times\left\{\begin{array}{l} p\left[g^{\mathrm{V}}(\alpha,n-1) - g^{\mathrm{V}}(\alpha,n-2)\right] + (n-1)g^{\mathrm{V}}(\alpha,n-2) \\ + \left[1-(n-1)\right]pg^{\mathrm{V}}(\alpha,n-2) \end{array}\right\} \\
&+ (1-p)^{n-2}\left\{p\left[g^{\mathrm{V}}(\alpha,1) - g^{\mathrm{V}}(\alpha,0)\right] + g^{\mathrm{V}}(\alpha,0) + \left[(n-1)-1\right]pg^{\mathrm{V}}(\alpha,1)\right\} \\
&+ \sum_{k=2}^{n-3}\binom{n-1}{k}p^k(1-p)^{n-1-k}g^{\mathrm{V}}(\alpha,k) \\
={}& p\left\{p^{n-2}\left[g^{\mathrm{V}}(\alpha,n-1) - g^{\mathrm{V}}(\alpha,n-2)\right] + (1-p)^{n-2}\left[g^{\mathrm{V}}(\alpha,1) - g^{\mathrm{V}}(\alpha,0)\right]\right\} \\
&+ p^{n-2}\left[(n-1)-(n-2)p\right]g^{\mathrm{V}}(\alpha,n-2) \\
&+ (1-p)^{n-2}\left[g^{\mathrm{V}}(\alpha,0) + (n-2)pg^{\mathrm{V}}(\alpha,1)\right] \\
&+ \sum_{k=2}^{n-3}\binom{n-1}{k}p^k(1-p)^{n-1-k}g^{\mathrm{V}}(\alpha,k).
\end{aligned}
$$

Then, $h^{\mathrm{V}}(\hat{p},\alpha,n) = 0$ is equivalent to

$$
\begin{aligned}
&\hat{p}^{n-2}\left[g^{\mathrm{V}}(\alpha,n-1) - g^{\mathrm{V}}(\alpha,n-2)\right] + (1-\hat{p})^{n-2}\left[g^{\mathrm{V}}(\alpha,1) - g^{\mathrm{V}}(\alpha,0)\right] \\
&\qquad = -\frac{1}{\hat{p}}\times\left\{\begin{array}{l} \hat{p}^{n-2}\left[(n-1)-(n-2)\hat{p}\right]g^{\mathrm{V}}(\alpha,n-2) \\ + (1-\hat{p})^{n-2}\left[g^{\mathrm{V}}(\alpha,0) + (n-2)\hat{p}g^{\mathrm{V}}(\alpha,1)\right] \\ + \displaystyle\sum_{k=2}^{n-3}\binom{n-1}{k}\hat{p}^k(1-\hat{p})^{n-1-k}g^{\mathrm{V}}(\alpha,k) \end{array}\right\}.
\end{aligned} \tag{15}
$$

Differentiation of h^V with respect to p and evaluating at $p = \hat{p}$ yields

$$
\begin{aligned}
\frac{\partial h^{\mathrm{V}}}{\partial p}(\hat{p},\alpha,n) &= (n-1)\sum_{k=0}^{n-2}\binom{n-2}{k}\hat{p}^k(1-\hat{p})^{n-2-k}\left[g^{\mathrm{V}}(\alpha,k+1) - g^{\mathrm{V}}(\alpha,k)\right] \\
&= (n-1)\sum_{k=1}^{n-3}\binom{n-2}{k}\hat{p}^k(1-\hat{p})^{n-2-k}\left[g^{\mathrm{V}}(\alpha,k+1) - g^{\mathrm{V}}(\alpha,k)\right]
\end{aligned}
$$

$$+ (n-1) \times \left\{ \begin{array}{l} \hat{p}^{n-2} \left[g^{\mathrm{V}}(\alpha, n-1) - g^{\mathrm{V}}(\alpha, n-2) \right] \\ + (1-\hat{p})^{n-2} \left[g^{\mathrm{V}}(\alpha, 1) - g^{\mathrm{V}}(\alpha, 0) \right] \end{array} \right\}.$$

By using (15), we obtain

$$\begin{aligned} \frac{\partial h^{\mathrm{V}}}{\partial \hat{p}}(\hat{p}, \alpha, n) &= (n-1) \sum_{k=1}^{n-3} \binom{n-2}{k} \hat{p}^k (1-\hat{p})^{n-2-k} \left[g^{\mathrm{V}}(\alpha, k+1) - g^{\mathrm{V}}(\alpha, k) \right] \\ &\quad - \frac{n-1}{\hat{p}} \times \left\{ \begin{array}{l} \hat{p}^{n-2} \left[(n-1) - (n-2)\hat{p} \right] g^{\mathrm{V}}(\alpha, n-2) \\ + (1-\hat{p})^{n-2} \left[g^{\mathrm{V}}(\alpha, 0) + (n-2)\hat{p} g^{\mathrm{V}}(\alpha, 1) \right] \\ + \sum_{k=2}^{n-3} \binom{n-1}{k} \hat{p}^k (1-\hat{p})^{n-1-k} g^{\mathrm{V}}(\alpha, k) \end{array} \right\} \\ &= -\frac{n-1}{\hat{p}} \times \Omega, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \Omega \equiv & \sum_{k=1}^{n-3} \binom{n-2}{k} \hat{p}^{k+1} (1-\hat{p})^{n-2-k} \left[g^{\mathrm{V}}(\alpha, k) - g^{\mathrm{V}}(\alpha, k+1) \right] \\ & + \hat{p}^{n-2} \left[(n-1) - \hat{p}(n-2) \right] g^{\mathrm{V}}(\alpha, n-2) + (1-\hat{p})^{n-2} \left[g^{\mathrm{V}}(\alpha, 0) + (n-2)\hat{p} g^{\mathrm{V}}(\alpha, 1) \right] \\ & + \sum_{k=2}^{n-3} \binom{n-1}{k} \hat{p}^k (1-\hat{p})^{n-1-k} g^{\mathrm{V}}(\alpha, k). \end{aligned}$$

Note here that

$$\begin{aligned} \Omega = & \sum_{k=2}^{n-3} \binom{n-2}{k} \hat{p}^{k+1} (1-\hat{p})^{n-2-k} g^{\mathrm{V}}(\alpha, k) - \sum_{k=1}^{n-4} \binom{n-2}{k} \hat{p}^{k+1} (1-\hat{p})^{n-2-k} g^{\mathrm{V}}(\alpha, k+1) \\ & + \hat{p}^{n-2} \left[(n-1) - \hat{p}(n-2) - (1-\hat{p}) \binom{n-2}{n-3} \right] g^{\mathrm{V}}(\alpha, n-2) \\ & + \hat{p}(1-\hat{p})^{n-3} \left[(1-\hat{p})(n-2) + \hat{p} \binom{n-2}{1} \right] g^{\mathrm{V}}(\alpha, 1) \\ & + (1-\hat{p})^{n-2} g^{\mathrm{V}}(\alpha, 0) + \sum_{k=2}^{n-3} \binom{n-1}{k} \hat{p}^k (1-\hat{p})^{n-1-k} g^{\mathrm{V}}(\alpha, k) \\ = & \sum_{k=2}^{n-3} \binom{n-2}{k} \hat{p}^{k+1} (1-\hat{p})^{n-2-k} g^{\mathrm{V}}(\alpha, k) - \sum_{k=2}^{n-3} \binom{n-2}{k-1} \hat{p}^k (1-\hat{p})^{n-1-k} g^{\mathrm{V}}(\alpha, k) \\ & + \hat{p}^{n-2} \left\{ \left[(n-1) - \binom{n-2}{n-3} \right] + \hat{p} \left[-(n-2) + \binom{n-2}{n-3} \right] \right\} g^{\mathrm{V}}(\alpha, n-2) \end{aligned}$$

$$
\begin{aligned}
&+\hat{p}(1-\hat{p})^{n-3}\left\{(n-2)+\hat{p}\left[\binom{n-2}{1}-(n-2)\right]\right\}g^{\mathrm{V}}(\alpha,1)\\
&+(1-\hat{p})^{n-2}g^{\mathrm{V}}(\alpha,0)+\sum_{k=2}^{n-3}\binom{n-1}{k}\hat{p}^{k}(1-\hat{p})^{n-1-k}g^{\mathrm{V}}(\alpha,k)\\
=\,&\hat{p}^{n-2}\left\{[(n-1)-(n-2)]+[-(n-2)+(n-2)]\,\hat{p}\right\}g^{\mathrm{V}}(\alpha,n-2)\\
&+\hat{p}(1-\hat{p})^{n-3}\left\{(n-2)+[(n-2)-(n-2)]\,\hat{p}\right\}g^{\mathrm{V}}(\alpha,1)+(1-\hat{p})^{n-2}g^{\mathrm{V}}(\alpha,0)\\
&+\sum_{k=2}^{n-3}\left[\begin{array}{l}\binom{n-1}{k}\hat{p}^{k}(1-\hat{p})^{n-1-k}-\binom{n-2}{k-1}\hat{p}^{k}(1-\hat{p})^{n-1-k}\\ +\binom{n-2}{k}\hat{p}^{k+1}(1-\hat{p})^{n-2-k}\end{array}\right]g^{\mathrm{V}}(\alpha,k)\\
=\,&\hat{p}^{n-2}g^{\mathrm{V}}(\alpha,n-2)+(1-\hat{p})^{n-2}g^{\mathrm{V}}(\alpha,0)+(n-2)\hat{p}(1-\hat{p})^{n-3}g^{\mathrm{V}}(\alpha,1)\\
&+\sum_{k=2}^{n-3}\left[\begin{array}{l}\binom{n-1}{k}\hat{p}^{k}(1-\hat{p})^{n-1-k}-\frac{k}{n-1}\binom{n-1}{k}\hat{p}^{k}(1-\hat{p})^{n-1-k}\\ +\binom{n-2}{k}\hat{p}^{k+1}(1-\hat{p})^{n-2-k}\end{array}\right]g^{\mathrm{V}}(\alpha,k)\\
=\,&\hat{p}^{n-2}g^{\mathrm{V}}(\alpha,n-2)+(1-\hat{p})^{n-2}g^{\mathrm{V}}(\alpha,0)+(n-2)\hat{p}(1-\hat{p})^{n-3}g^{\mathrm{V}}(\alpha,1)\\
&+\sum_{k=2}^{n-3}\left[\frac{n-1-k}{n-1}\binom{n-1}{k}\hat{p}^{k}(1-\hat{p})^{n-1-k}+\binom{n-2}{k}\hat{p}^{k+1}(1-\hat{p})^{n-2-k}\right]g^{\mathrm{V}}(\alpha,k)\\
=\,&\hat{p}^{n-2}g^{\mathrm{V}}(\alpha,n-2)+(1-\hat{p})^{n-2}g^{\mathrm{V}}(\alpha,0)+(n-2)\hat{p}(1-\hat{p})^{n-3}g^{\mathrm{V}}(\alpha,1)\\
&+\sum_{k=2}^{n-3}\left[\binom{n-2}{k}\hat{p}^{k}(1-\hat{p})^{n-1-k}+\frac{\hat{p}}{1-\hat{p}}\binom{n-2}{k}\hat{p}^{k}(1-\hat{p})^{n-1-k}\right]g^{\mathrm{V}}(\alpha,k)\\
=\,&\hat{p}^{n-2}g^{\mathrm{V}}(\alpha,n-2)+(1-\hat{p})^{n-2}g^{\mathrm{V}}(\alpha,0)+(n-2)\hat{p}(1-\hat{p})^{n-3}g^{\mathrm{V}}(\alpha,1)\\
&+\frac{1}{1-\hat{p}}\sum_{k=2}^{n-3}\binom{n-2}{k}\hat{p}^{k}(1-\hat{p})^{n-1-k}g^{\mathrm{V}}(\alpha,k)\\
=\,&\binom{n-2}{n-2}\hat{p}^{n-2}g^{\mathrm{V}}(\alpha,n-2)+\binom{n-2}{0}(1-\hat{p})^{n-2}g^{\mathrm{V}}(\alpha,0)\\
&+\binom{n-2}{1}\hat{p}(1-\hat{p})^{n-3}g^{\mathrm{V}}(\alpha,1)+\sum_{k=2}^{n-3}\binom{n-2}{k}\hat{p}^{k}(1-\hat{p})^{n-2-k}g^{\mathrm{V}}(\alpha,k)\\
=\,&\sum_{k=0}^{n-2}\binom{n-2}{k}\hat{p}^{k}(1-\hat{p})^{n-2-k}g^{\mathrm{V}}(\alpha,k)\\
=\,&h^{\mathrm{V}}(\hat{p},\alpha,n-1).
\end{aligned}
$$

By using this, we rewrite (16) as

$$\frac{\partial h^{\mathrm{V}}}{\partial \hat{p}}(\hat{p},\alpha,n) = -\frac{n-1}{\hat{p}} \times h^{\mathrm{V}}(\hat{p},\alpha,n-1).$$

Since $h^{\mathrm{V}}(\hat{p},\alpha,n-1) > 0$ by Step 1, we have $\frac{\partial h^{\mathrm{V}}}{\partial p}(\hat{p},\alpha,n) < 0$. □

B.2 Proof of statement (ii)

Note that $\lim_{p\downarrow 0} h^{\mathrm{V}}(p,\alpha,n) = \lim_{p\downarrow 0} h^{\mathrm{V}}(p,\alpha,n+1) = \alpha^{\alpha} > 0$ and both $h^{\mathrm{V}}(\,\cdot\,,\alpha,n)$ and $h^{\mathrm{V}}(\,\cdot\,,\alpha,n+1)$ are continuous in p. Moreover, $p^{\mathrm{V}}(\alpha,n+1) < 1$ implies $h^{\mathrm{V}}(p^{\mathrm{V}}(\alpha,n+1),\alpha,n+1) = 0$. Recall the proof of statement (i) of Theorem 2. Then, $\frac{\partial h^{\mathrm{V}}}{\partial p}(p^{\mathrm{V}}(\alpha,n+1),\alpha,n+1) < 0$. If there is $\hat{p} \in]0,1[$ with $h^{\mathrm{V}}(\hat{p},\alpha,n) = 0$, then $\frac{\partial h^{\mathrm{V}}}{\partial p}(\hat{p},\alpha,n) < 0$. Thus, in order to show that $p^{\mathrm{V}}(\alpha,n) > p^{\mathrm{V}}(\alpha,n+1)$ whenever $p^{\mathrm{V}}(\alpha,n+1) < 1$, it is sufficient to show that $h^{\mathrm{V}}(p,\alpha,n+1) = 0$ implies $h^{\mathrm{V}}(p,\alpha,n) > 0$.

Suppose that $h^{\mathrm{PE}}(p,\alpha,n+1) = 0$. Let $\Delta^{\mathrm{V}}(p,\alpha,n) \equiv 2 \times (1-p)h^{\mathrm{V}}(p,\alpha,n) - h^{\mathrm{V}}(p,\alpha,n+1)$. Since $\Delta^{\mathrm{V}}(p,\alpha,n) > 0$ implies $h^{\mathrm{V}}(p,\alpha,n) > 0$, we now show that $\Delta^{\mathrm{V}}(p,\alpha,n) > 0$. Let $k^* > 0$ be such that $g^{\mathrm{V}}(\alpha,k^*) < 0$ and $g^{\mathrm{V}}(\alpha,k^*-1) \geq 0$. Since $h^{\mathrm{PE}}(p,\alpha,n+1) = 0$, Lemmas 2–4 ensures that such k^* exists. Moreover, it holds the followings: (i) for each $k \in \{0,\ldots,k^*-2\}$, $g^{\mathrm{V}}(\alpha,k) > 0$; (ii) $g^{\mathrm{V}}(\alpha,k^*-1) \geq 0$; and (iii) for each $k \in \{k^*,\ldots,n\}$, $g^{\mathrm{V}}(\alpha,k) < 0$. Then, using similar arguments as those employed in the proof of statement (ii) of Theorem 1, we can prove that $\Delta^{\mathrm{V}}(p,\alpha,n) > 0$. Moreover, we observe that $\lim_{k\to\infty} p^{\mathrm{V}}(\alpha,k) = 0$ because the sequence $\{p^{\mathrm{V}}(\alpha,k)\}_{k\geq 2}$ is monotonically decreasing and bounded by 0 from below. □

C Appendix: Proof of Theorem 3

We proceed in three steps.

Step 1: For each $\alpha \in]0,1[$, $\frac{\partial h^{\mathrm{V}}}{\partial p}(p,\alpha,2) < 0$. Note that

$$\begin{aligned}
\frac{\partial h^{\mathrm{V}}}{\partial p}(p,\alpha,2) &= g^{\mathrm{V}}(\alpha,1) - g^{\mathrm{V}}(\alpha,0) \\
&= \frac{2\alpha^{\alpha}}{1+\alpha} - 1 - \alpha^{\alpha} \\
&= \frac{2\alpha^{\alpha} - (1+\alpha) - \alpha^{\alpha}(1+\alpha)}{1+\alpha} \\
&= \frac{(\alpha^{\alpha}-1) - \alpha(1+\alpha^{\alpha})}{1+\alpha}.
\end{aligned}$$

Since $\alpha \in]0,1[$, we have $\alpha^\alpha - 1 < 0$, which implies $\frac{\partial h^{\mathrm{V}}}{\partial p}(p,\alpha,2) < 0$.

Step 2: For each $\boldsymbol{\alpha \in]0,1[,\ 1+\alpha > 2^\alpha}$. Let $\varphi(\alpha) \equiv 1+\alpha$ and $\psi(\alpha) \equiv 2^\alpha$. Note that $\lim_{\alpha\downarrow 0}\varphi(\alpha) = \lim_{\alpha\downarrow 0}\psi(\alpha) = 1$ and $\lim_{\alpha\uparrow 1}\varphi(\alpha) = \lim_{\alpha\uparrow 1}\psi(\alpha) = 2$. Then, $\varphi'(\alpha) = 1 > 0$, $\varphi''(\alpha) = 0$, $\psi'(\alpha) = 2^\alpha \ln 2 > 0$, and $\psi''(\alpha) = 2^\alpha (\ln 2)^2 > 0$. Since both φ and ψ are continuous, we have that for each $\alpha \in]0,1[$, $\varphi(\alpha) > \psi(\alpha)$, that is, $1+\alpha > 2^\alpha$.

Step 3: Concluding. Let $\alpha \in]0,1[$. As we have shown in proof of statement (i) of Theorem 1, $\frac{\partial h^{\mathrm{PE}}}{\partial p}(p,\alpha,2) < 0$. Moreover, by Step 1, $\frac{\partial h^{\mathrm{V}}}{\partial p}(p,\alpha,2) < 0$. Thus, $h^{\mathrm{PE}}(p,\alpha,2) - h^{\mathrm{V}}(p,\alpha,2) > 0$ implies that if $p^{\mathrm{V}}(\alpha,2) = 1$, then $p^{\mathrm{PE}}(\alpha,2) = 1$; otherwise, $p^{\mathrm{PE}}(\alpha,2) > p^{\mathrm{V}}(\alpha,2)$. We now show that $h^{\mathrm{PE}}(p,\alpha,2) - h^{\mathrm{V}}(p,\alpha,2) > 0$. Then,

$$\begin{aligned}
h^{\mathrm{PE}}(p,\alpha,2) - h^{\mathrm{V}}(p,\alpha,2) &= (1-p)[g^{\mathrm{PE}}(\alpha,0) - g^{\mathrm{V}}(\alpha,0)] + p[g^{\mathrm{PE}}(\alpha,1) - g^{\mathrm{V}}(\alpha,1)] \\
&= (1-p)(\alpha^\alpha - \alpha^\alpha) + p\left[\frac{2\alpha^\alpha}{2^\alpha} - 1 - \left(\frac{2\alpha^\alpha}{1+\alpha} - 1\right)\right] \\
&= p2\alpha^\alpha\left[\frac{(1+\alpha) - 2^\alpha}{2^\alpha(1+\alpha)}\right].
\end{aligned}$$

By Step 2, $(1+\alpha) - 2^\alpha > 0$. This implies $h^{\mathrm{PE}}(p,\alpha,2) - h^{\mathrm{V}}(p,\alpha,2) > 0$. □

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