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Strategic justification in claims problems: Procedurally fair and multilateral bargaining game*

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Abstract

In a claims problem, a group of agents have claims on the liquidation value of a bankrupt firm, but there is not enough to honor the totality of the claims. A central rule for claims problems is the so-called constrained equal awards rule. For a strategic justification of the rule, we propose a procedurally fair and multilateral bargaining game. We show that for each claims problem, the awards vector chosen by the rule is the unique subgame perfect equilibrium outcome of the game.

JEL Classification: C71; C72; D63

Key words: Nash program; Strategic justification; Claims problem; Constrained equal awards rule

1 Introduction

In a claims problem, a group of agents have claims on the liquidation value of a bankrupt firm, which we call the "endowment." The endowment is not sufficient to honor the totality of the claims.[1] How should it be allocated? A "rule" is a single-valued mapping that associates, with each claims problem, an allocation of the endowment satisfying non-negativity, claims boundedness, and efficiency. We call such an allocation an "awards vector." As a central rule for claims problems, we consider the so-called constrained equal awards (CEA) rule. The CEA rule satisfies a number of desirable properties, and the rule has been characterized in multiple ways, reviewed in Thomson (2019).[2] Our purpose is

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[1] For a comprehensive survey on claims problems, see Thomson (2019).

[2] See, for example, Theorem 6.4 at pp 144 of Thomson (2019), which is due to Dagan (1996).

to develop a strategic justification of the rule.

Nash (1953) initiates the study of strategic justifications of cooperative solutions for bargaining problems through non-cooperative procedures. Specifically, he provides a strategic justification for the Nash bargaining solution (Nash, 1950) through the so-called "Nash demand game." This line of research is known as the Nash program.[3] The CEA rule corresponds to the Nash bargaining solution in a special class of bargaining problems, as discussed in Remark 3. In this sense, we follow the original work of Nash (1953) as a part.

For each claims problem, we propose a game associated with the problem for which the awards vector chosen by the rule is the unique subgame perfect equilibrium (SPE) outcome of the game. Our game, as well as other games that have been offered to provide non-cooperative foundations of rules for claims problems, such as in Tsay and Yeh (2019), requires that claims be known to the one in charge of the design task.[4] As a justification, by citing Serrano (2005), "*the community of n agents involved in the problem at hand can be the one in charge of the design task*" (pp 8). In addition, we assume that the community in charge of the design task has the ability to impose the outcome of the game once played.[5] Since a game is designed by the claimants and the outcome once made is enforced, it is desirable that the game be "procedurally fair" (claimants be treated equally) and "multilateral" (all claimants negotiate simultaneously). If some claimant is not treated equally, for example, the agent with a maximum claim has a power to select his assignment, then the other claimant may not want to participate in the game. However, since the community of claimants is in charge of the design task, the

[3]For a comprehensive survey on the Nash program, see Serrano (2005).

[4]By contrast, in implementation theory, we usually assume that the designer does not know the agents' private information, and the aim of the theory is to provide a game in which the set of equilibrium outcomes is equal to the outcome chosen by a rule. If the agents' private information is known to the designer and the goal of the Nash program were only seen as the same aim of implementation theory, we can achieve this by the trivial game: if all agents agree on the outcome chosen by the rule, we select it; otherwise, we select a bad outcome (see Proposition 1 of Bergin and Duggan, 1999). However, from the viewpoint of strategic justifications, this game does not shed any light on the meaning of the specific rule. For a discussion on the relationship between the Nash program and implementation theory, see Serrano (2005).

[5]This assumption is discussed by Nash (1953). "$\cdots$ *we must assume there is an adequate mechanism for forcing the players to stick to their threats and demands once made, for one to enforce the bargain once agreed*" (pp 130).

claimants already agree with the procedure when the game is played. If at some stage of the game, some claimant does not negotiate with the other claimants at all, then he may complain about the final allocation even if before participating, he knows that the outcome once made is enforced. Then, we incorporate the two features into our games. While Li and Ju (2016) and Tsay and Yeh (2019) also study the strategic justification of the CEA rule, their games do not have at least one of the two features. For a detailed discussion, see Subsection 1.1.

Our game is as follows: At each period t, each claimant proposes a pair consisting of an awards vector and a permutation. If some awards vector is proposed by more than one claimant, then the awards vector which receives the highest number of votes is chosen as temporary awards vector. The components of this temporary awards vector are subjected to the composition of the reported permutations, and then the game ends.[6] If no two claimants propose the same awards vector, the game proceeds to the next period $t + 1$ and we repeat the above process. The formal definition of the above game is proposed in Section 3.

Our game resembles the simultaneous-offers bargaining game analyzed in Chatterjee and Samuelson (1990) in the sense that all agents simultaneously make proposals and if they cannot reach an agreement in the current period, they can renegotiate in the next period.[7] The simultaneous-offers bargaining game is the Nash demand game with renegotiation. In the games proposed by Li and Ju (2016) and Tsay and Yeh (2019), most claimants do not have a chance to renegotiate (see Subsection 1.1), but in our game, all claimants can do so if they cannot reach an agreement.

We show that for each claims problem, the awards vector chosen by the CEA rule achieved at period 1 is supported as an SPE outcome of the game associated with the problem (Proposition 1). In addition, for each claims problem, any SPE outcome of the

[6] The permutation idea is proposed in Thomson (2005) for the allocation of a social endowment of infinitely divisible resources and exploited by Doğan (2016), Hagiwara (2019), and others. Chang and Hu (2008), Hayashi and Sakai (2009), Moreno-Ternero, Tsay, and Yeh (2018), and Tsay and Yeh (2019) apply the idea of letting each agent report a permutation as a strategy into their games to exchange the order of claimants not for exchanging an allocation.

[7] Note that, although Chatterjee and Samuelson (1990) consider the case of only two agents, our game is applicable to any number of agents.

game associated with the problem is the awards vector chosen by the CEA rule achieved at period 1 (Proposition 2). By Propositions 1 and 2, for each claims problem, the awards vector chosen by the CEA rule achieved at period 1 is the unique SPE outcome of the game associated with the problem (Theorem 1).[8]

Our results have two applications, one to bargaining problems and one to coalitional problems. For detailed discussions of these applications, see Remarks 3 and 4.

Correspondences between rules for claims problems and bargaining solutions have been identified (see, for example, Theorem 14.1 at pp 361 of Thomson, 2019).[9] In particular, the CEA rule corresponds to the Nash bargaining solution.[10] From this correspondence, our Theorem 1 implies that for the bargaining problem associated with a claims problem, our game provides a strategic justification of the Nash bargaining solution.

This implication resembles a result of Anbarci and Boyd III (2011). They propose a game, which is a variant of the Nash demand game, and they call it the "Simultaneous Procedure." They show that for each two-person bargaining problem, the outcome chosen by the Nash bargaining solution is the unique Nash equilibrium outcome of their game. In contrast to their game, our game is applicable to any number of agents and it allows renegotiation. In addition, as a special kind of bargaining problem, our results are applicable to the "Divide-the-Dollar problem," where the endowment is equal to 1 and each agent's claim is at least 1.[11]

Moreover, correspondences between rules for claims problems and solutions to coalitional problems have been identified (see, for example, Theorem 14.2 at pp 373 of Thomson, 2019). In particular, the CEA rule corresponds to the Dutta-Ray solution (Dutta and Ray, 1989).[12] From this correspondence, our Theorem 1 implies that for the coalitional

[8]While our game is similar to the simultaneous-offers bargaining game of Chatterjee and Samuelson (1990) as we discussed, our result is contrast to the result of them where all outcomes including disagreement are supported as the SPE outcomes of their game.

[9]If for each claims problem, the outcome chosen by the rule coincides with the outcome chosen by the solution when applied to the associated bargaining problem, then we say that the rule corresponds to the bargaining solution. This definition of correspondence is also applied to coalitional problems.

[10]This result is proposed by Dagan and Volij (1993).

[11]For strategic justifications in Divide-the-Dollar problems, see, for example, Anbarci (2001), Rachmilevitch (2017), and Karagözoğlu and Rachmilevitch (2018).

[12]For solutions in coalitional problems, strategic justifications have been discussed. For example, Pérez-Castrillo and Wettstein (2001) propose the "bidding mechanism" in which for each coalitional problem, the SPE outcomes of this mechanism coincide with the vector of the Shapley value payoffs.

problem associated with a claims problem, our game provides a strategic justification of the Dutta-Ray solution.

1.1 Related literature

Strategic justifications of the CEA rule for claims problems have been derived by Li and Ju (2016) and Tsay and Yeh (2019).[13]

Li and Ju (2016) propose the following n-stage game, where n is the number of claimants. Claimants are numbered following the reverse order of claims. In Stage 1, the agent whose claim is largest, claimant n, divides the endowment as proposal. In Stage $k \in \{2, \ldots, n\}$, claimant $k-1$ chooses a component of the proposal as their payoffs.[14] Agents with lower claims are given priority to choose early on in the game, but claimant n is given the power to make the division. Therefore, in this game, claimants are not treated equally. In addition, at each stage of their game, only two claimants negotiate, so that the game is bilateral even in the case of more than two claimants. Furthermore, any claimant who negotiated at a stage of their game with claimant n cannot negotiate anymore.

Tsay and Yeh (2019) propose the following three-stage game for the CEA rule. In Stage 1, each claimant announces a pair consisting of an awards vector and a permutation. The composition of the reported permutations selects a claimant as coordinator. If all claimants, except for the coordinator, announce the same awards vector, then this awards vector is the proposal; otherwise, the awards vector announced by the coordinator is the proposal. In Stage 2, the coordinator either accepts or rejects the proposal. If he accepts it, the proposal is the outcome. If he rejects it, he selects one claimant to negotiate awards for the two of them[15]; all the others receive their awards as specified in the proposal. That is, in the first stage, all claimants are given the power to choose the temporary

[13] Strategic justifications of other rules have been also studied. For the family of f-just rules, see Dagan, Serrano, and Volij (1997) and Chang and Hu (2008). For the constrained equal losses rule, see Li and Ju (2016) and Tsay and Yeh (2019). For the Talmud rule, see Li and Ju (2016), Moreno-Ternero, Tsay, and Yeh (2018), and Tsay and Yeh (2019). For the proportional rule, see Tsay and Yeh (2019).

[14] This bilateral negotiation procedure is similar to the games of Chae and Yang (1988) and Sonn (1992).

[15] For the bilateral negotiation game for the CEA rule, see Tsay and Yeh (2019).

awards vector, but in the second stage, only the coordinator is given the power to reject a component of the proposal and to choose an claimant to negotiate on the final awards for them. Therefore, since no claimant except for the coordinator has the power to reject components of the proposal, claimants are not treated equally. Moreover, in their game, only the coordinator and the claimant selected by the coordinator at Stage 2 have one chance to renegotiate.

By contrast, in our game, all claimants are treated equally; they negotiate multilaterally; and they have chances to renegotiate.

2 The model

Let $N = \{1, \ldots, n\}$ be the set of agents with $n \geq 2$. Each agent $i \in N$ has a claim on a resource, $c_i \in \mathbb{R}_+$. Claimants are numbered so that $c_1 \leq \cdots \leq c_n$. Let $c \equiv (c_1, \ldots, c_n)$ be a claims vector. There is an endowment E of the resource. The endowment is insufficient to honor the totality of the claims. Using $\mathbb{R}^N_+$ for the cross-product of n copies of $\mathbb{R}_+$ indexed by the members of N, a **claims problem** is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_{++}$ such that $E \leq \sum_{i \in N} c_i$. Let $\mathcal{C}^N$ denote the domain of all claims problems.

An **awards vector** for the claims problem $(c, E) \in \mathcal{C}^N$ is a vector $a \equiv (a_1, \ldots, a_n) \in \mathbb{R}^N_+$ (i.e., non-negativity) such that, for each $i \in N$, $a_i \leq c_i$ (i.e., claims boundedness) and $\sum_{i \in N} a_i = E$ (i.e., efficiency). Let $A(c, E) = \{a \in \mathbb{R}^n_+ \mid \text{ for each } i \in N, a_i \leq c_i \text{ and } \sum_{i \in N} a_i = E\}$ be the set of awards vectors of the problem $(c, E) \in \mathcal{C}^N$. A division rule, or simply a **rule**, is a single-valued mapping which associates, with each problem $(c, E) \in \mathcal{C}^N$, an awards vector $a \in A(c, E)$.

The following is central to our study:[16]

Constrained equal awards rule, CEA**:** For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, $CEA_i(c, E) \equiv \min\{c_i, \lambda\}$, where $\lambda \in \mathbb{R}_+$ is chosen so as to satisfy efficiency.

[16] For other important rules for claims problems, see, for example, Ch.2 of Thomson (2019).

3 The game for a strategic justification of the CEA rule

For a strategic justification of the CEA rule, we consider the following game starting at period 1. At each period t, each claimant proposes a pair consisting of an awards vector and a permutation. If some awards vector is proposed by more than one claimant, then the awards vector which receives the highest number of votes is chosen as temporary awards vector. If at least two awards vectors receive the highest number of votes, then the awards vector proposed by the claimant who has the lowest index among the claimants who announce one of these awards vectors is chosen as temporary awards vector. The components of this awards vector are subjected to the composition of the reported permutations, and then the game ends. Note that no matter what the permutations reported by the other claimants are and no matter what order of permutations for the composition is, each claimant can assign any component of the temporary awards vector to himself by proposing an appropriate permutation. A claimant's payoff is the discounted present value of the minimum of his claim or his component of resulting allocation adjusted so as to satisfy claims boundedness. If no two claimants propose the same awards vector, the game proceeds to the next period $t+1$ and we repeat the above process. If the claimants cannot reach an agreement permanently, then the negotiation breaks down.

To define the game formally, let us introduce some notation. A **permutation** $\pi : N \to N$ is a one-to-one function from N to N. Let Π be the set of permutations. Let $\delta \in (0,1)$ be the claimants' **common discount factor**.

Let $(c, E) \in \mathcal{C}^N$ be given. The game $\Gamma(c, E)$ is as follows:

1. At period t, each claimant i proposes a pair $(a^i, \pi^i) \in A(c, E) \times \Pi$.

2. If for some $a \in A(c, E)$, $|\{i \in N \mid a^i = a\}| \geq 2$, we select $a^{\hat{i}}$, where $\hat{i} = \min\{i' \in N \mid a^{i'} \in \arg\max_{a \in A} |\{i \in N \mid a^i = a\}|\}$.

   The components of the awards vector $a^{\hat{i}}$ are exchanged by according to the composition $\pi^* \equiv \pi^n \circ \cdots \circ \pi^1$, and the game ends. This outcome at period t is denoted by $[a^{\hat{i}}_{\pi^*}, t]$, where $a^{\hat{i}}_{\pi^*} = (a^{\hat{i}}_{\pi^*(1)}, \ldots, a^{\hat{i}}_{\pi^*(n)})$. When claimant i obtains $a^{\hat{i}}_{\pi^*(i)}$ at period

t, his payoff is $\delta^{t-1} \min\{c_i, a^{\hat{i}}_{\pi^*(i)}\}$.

3. If for each $a \in A$, $|\{i \in N \mid a^i = a\}| \leq 1$, then the game proceeds to the next period $t+1$ and we repeat the above process.

   If the claimants cannot reach an agreement permanently, disagreement occurs and then each claimant obtains a payoff of zero.

Regarding our game, there are the following two remarks.

Remark 1. (Tie-breaking). We use a tie-breaking when at least two awards vectors receive the highest number of votes. Note that no matter what tie-breaking is used, our results hold. One may say that this game does not treat claimants equally.

To resolve this problem for procedural fairness, let each claimant additionally report another permutation and by according to the composition of these permutations, a claimant who chooses one awards vector in those receiving the highest number of votes is selected as tie-breaker. Then, the awards vector reported by the tie-breaker is selected as temporary awards vector. In this modified game, no matter what the permutations reported by the other claimants are, any claimant reporting one awards vector in those receiving the highest number of votes can be the tie-breaker by proposing an appropriate permutation. Therefore, this modified game is procedurally fair. Since the definition of our game and the proofs of Propositions 1 and 2 are simpler, we propose those in this paper.$\diamondsuit$

Remark 2. (Inefficient allocations). In our game, some final allocation after exchange may not satisfy efficiency. Chang and Hu (2008) also use inefficient allocations in their game. In their game, at the first stage, if some claimant reports a different awards vector from the awards vectors announced by the other claimants, then the coordinator, who is the first claimant selected by the composition of the reported permutations, gets a negative value and the other claimants gets nothing. In their game, inefficient allocations have an important role to have Nash equilibria of their game, but in our game, these are just selected so as to satisfy claims boundedness.$\diamondsuit$

In the following, we derive an SPE outcome of $\Gamma(c, E)$.

Proposition 1. *For each* $(c, E) \in \mathcal{C}^N$, $[CEA(c, E), 1]$ *is supported as an SPE outcome of* $\Gamma(c, E)$.

Proof. Let $\sigma^* \equiv (\sigma_1^*, \ldots, \sigma_n^*)$ be the strategy profile such that each claimant $i \in N$ always proposes $(CEA(c, E), \pi_{id})$, where π_{id} is the identity permutation i.e., for each $i \in N$, $\pi_{id}(i) = i$. The outcome under σ^* is $[CEA(c, E), 1]$. Then, claimant i's payoff is $CEA_i(c, E)$ $(\leq c_i)$. We prove that σ^* is an SPE of $\Gamma(c, E)$.

We use the one-shot deviation principle (see, for example, Fudenberg and Tirole, 1991): a strategy profile σ is an SPE if and only if no claimant gains by deviating from σ in a single action. Fix $i \in N$ and a positive integer t arbitrarily. Suppose that claimant i deviates from σ_i^* in a single action and proposes some $(a^i, \pi^i) \neq (CEA(c, E), \pi_{id})$ at period t. First, we consider the case $E < \sum_{i \in N} c_i$. The proof is divided into two cases, $n \geq 3$ and $n = 2$. Note that the proof for the case $n \geq 3$ is applied to that for the case $n = 2$ as a part, so that we first consider the case $n \geq 3$.

Case $n \geq 3$**.** For any (a^i, π^i), $\arg\max_{a \in A} |\{i' \in N \mid a^{i'} = a\}| = \{CEA(c, E)\}$. If claimant i proposes (a^i, π^i), then since $\pi_{id} \circ \cdots \circ \pi^i \circ \cdots \pi_{id} = \pi^i$, he obtains $CEA_{\pi^i(i)}(c, E)$. Thus, to see that he cannot gain by deviating from σ_i^*, we prove that for each $\pi^i \in \Pi$, $\delta^{t-1} \min\{c_i, CEA_{\pi^i(i)}(c, E)\} \leq \delta^{t-1} CEA_i(c, E)$. It suffices to prove that for each $j \in N$, $\min\{c_i, CEA_j(c, E)\} \leq CEA_i(c, E)$.

By the definition of CEA, for each $j \in N$, $CEA_j(c, E) = \min\{c_j, \lambda\}$, where $\lambda \in \mathbb{R}_+$ is chosen so as to satisfy $\sum_{j \in N} CEA_j(c, E) = E$. If $CEA_i(c, E) = c_i$, then we immediately obtain that for each $j \in N$, $\min\{c_i, CEA_j(c, E)\} \leq CEA_i(c, E) (= c_i)$. If $CEA_i(c, E) < c_i$, then $\lambda = CEA_i(c, E) (< c_i)$. Since $\lambda < c_i$, we have that for each $j \in N$, $\min\{c_i, CEA_j(c, E)\} = \min\{c_i, \min\{c_j, \lambda\}\} = \min\{c_j, \lambda\} \leq \lambda = CEA_i(c, E)$. Therefore, claimant i cannot gain by deviating from σ_i^*.

Case $n = 2$**.** If claimant i proposes (a^i, π^i) such that $a^i = CEA(c, E)$ and $\pi^i \neq \pi_{id}$, then $\arg\max_{a \in A} |\{i' \in N \mid a^{i'} = a\}| = \{CEA(c, E)\}$. Therefore, since $\pi_{id} \circ \pi^i (= \pi^i \circ \pi_{id}) = \pi^i$ and $\pi^i \neq \pi_{id}$, claimant i obtains $CEA_j(c, E)$, where $j \neq i$. Analogously to the case $n \geq 3$, we obtain that $\delta^{t-1} \min\{c_i, CEA_j(c,E)\} \leq \delta^{t-1} CEA_i(c,E)$. Thus, claimant i cannot gain by deviating from σ_i^* if he proposes (a^i, π^i) such that $a^i = CEA(c,E)$ and $\pi^i \neq \pi_{id}$.

If claimant i proposes (a^i, π^i) such that $a^i \neq CEA(c,E)$, then for each $a \in A(c,E)$, $|\{i' \in N \mid a^{i'} = a\}| \leq 1$. Thus, the game proceeds to the next period $t+1$. Since claimant i follows σ_i^* at period $t+1$, his payoff is $\delta^t CEA_i(c,E)$. If claimant i does not deviate from σ_i^*, his payoff is $\delta^{t-1} CEA_i(c,E) (\geq \delta^t CEA_i(c,E))$. Therefore, he cannot gain by deviating from σ_i^*.

By the above discussion, in the case $\sum_{i \in N} c_i > E$, we have that σ^* is an SPE of $\Gamma(c,E)$. In the case $\sum_{i \in N} c_i = E$, since $A(c,E) = \{CEA(c,E)\}$, this case is proved by an analogous proof to that for $n \geq 3$. Therefore, $[CEA(c,E), 1]$ is supported as an SPE outcome of $\Gamma(c,E)$. □

The following is the uniqueness part of our strategic justification of the CEA rule.

Proposition 2. *For each $(c,E) \in \mathcal{C}^N$, any SPE outcome of $\Gamma(c,E)$ is $[CEA(c,E), 1]$.*

Proof. First, we show that for any $b \neq CEA(c,E)$, $[b,t]$ is not supported as an SPE outcome of $\Gamma(c,E)$. Suppose, by contradiction, that there exists an SPE σ of $\Gamma(c,E)$ whose outcome is $[b,t]$. We show that some claimant gains by deviating from σ. Let $\lambda \in \mathbb{R}^n_+$ be such that $\sum_{i \in N} \min\{c_i, \lambda\} = E$. Since $b \neq CEA(c,E)$, the proof is divided into the following three cases.

Case 1. For some $i^* \in N$, $b_{i^*} > c_{i^*}$.

Case 2. For each $i \in N$, $b_i \leq c_i$, and

2-1. for some $i^{**} \in N$ such that $c_{i^{**}} \leq \lambda$, $b_{i^{**}} < c_{i^{**}}$, or

2-2. for some $i^{***} \in N$ such that $c_{i^{***}} > \lambda$, $b_{i^{***}} \neq \lambda$.

We sequentially analyze each case.[17]

[17]When $\sum_{i \in N} c_i = E$, it suffices to consider Case 1 because of the following reason. When $\sum_{i \in N} c_i = E$, if for each $i \in N$, $b_i \leq c_i$, then $E = \sum_{i \in N} b_i \leq \sum_{i \in N} c_i = E$. This implies that for each $i \in N$, $b_i = c_i$, which contradicts the assumption that $b \neq CEA(c,E)$.

Case 1. Let $\hat{a}$ be the temporary awards vector before b is achieved at period t by exchange. For each $i \in N$, let π_σ^i be the permutation proposed by claimant i at period t under σ. That is, for each $i \in N$,

$$\hat{a}_{\pi_\sigma^n \circ \cdots \circ \pi_\sigma^1(i)} = b_i.$$

We show that for some $j \in N$, $b_j < \hat{a}_j (\leq c_j)$. Suppose, by contradiction, that for each $i \in N$, $\hat{a}_i \leq b_i$. If for some $k \in N$, $\hat{a}_k < b_k$, then $E = \sum_{i \in N} \hat{a}_i < \sum_{i \in N} b_i = E$, which is a contradiction. Thus, for each $i \in N$, $\hat{a}_i = b_i (= \hat{a}_{\pi_\sigma^n \circ \cdots \circ \pi_\sigma^1(i)})$. However, $c_{i^*} \geq \hat{a}_{i^*} = b_{i^*} > c_{i^*}$, which is also a contradiction. Consequently, for some $j \in N$, $\hat{a}_j > b_j (= \hat{a}_{\pi_\sigma^n \circ \cdots \circ \pi_\sigma^1(j)})$. Claimant j can assign $\hat{a}_j$ to himself by proposing an appropriate π^j $(\neq \pi_\sigma^j)$. Since $b_j < \hat{a}_j (\leq c_j)$, claimant j gains by deviating from σ_j, which contradicts the hypothesis that σ is an SPE of $\Gamma(c, E)$.

Case 2-1. We show that for some $j \in N$, $b_j > \lambda$. Suppose, by contradiction, that for each $i \in N$, $b_i \leq \lambda$. Since for each $i \in N$, $b_i \leq c_i$, then for each $k \in N$ such that $c_k \leq \lambda$, $b_k \leq c_k = CEA_k(c, E)$. In addition, for each $\ell \in N$ such that $c_\ell > \lambda$, $b_\ell \leq \lambda = CEA_\ell(c, E)$. Therefore, for each $i \in N$, $b_i \leq CEA_i(c, E)$. Since $b_{i^{**}} < c_{i^{**}} (= CEA_{i^{**}}(c, E))$, we have $\sum_{i \in N} b_i < \sum_{i \in N} CEA_i(c, E)$, which contradicts the assumption that $\sum_{i \in N} b_i = \sum_{i \in N} CEA_i(c, E) = E$. Thus, for some $j \in N$, $b_j > \lambda$. Claimant i^{**} can obtain b_j by proposing an appropriate $\pi^{i^{**}}$ $(\neq \pi_\sigma^{i^{**}})$. Since $b_j > \lambda \geq c_{i^{**}} > b_{i^{**}}$, claimant i^{**} gains by deviating from $\sigma_{i^{**}}$, which contradicts the hypothesis that σ is an SPE of $\Gamma(c, E)$.

Case 2-2. When $b_{i^{***}} < \lambda$, we show that for some $j \in N$, $b_j > \lambda$. Suppose, by contradiction, that for each $i \in N$, $b_i \leq \lambda$. Then, by the same proof as in Case 2-1, we obtain that for each $i \in N$, $b_i \leq CEA_i(c, E)$. Since $b_{i^{***}} < \lambda (= CEA_{i^{***}}(c, E))$, we have $\sum_{i \in N} b_i < \sum_{i \in N} CEA_i(c, E)$, which contradicts the assumption that $\sum_{i \in N} b_i = \sum_{i \in N} CEA_i(c, E) = E$. Thus, for some $j \in N$, $b_j > \lambda$. Claimant i^{***} can obtain b_j by proposing an appropriate $\pi^{i^{***}}$ $(\neq \pi_\sigma^{i^{***}})$. Since $b_j > \lambda \geq c_{i^{***}} > b_{i^{***}}$, claimant i^{***} gains by deviating from $\sigma_{i^{***}}$, which contradicts the hypothesis that σ is an

SPE of $\Gamma(c, E)$.

When $b_{i^{***}} > \lambda$, we show that for some $j \in N$ such that $c_j > \lambda$, $b_j < \lambda$. Suppose, by contradiction, that for each $i \in N$ such that $c_i > \lambda$, $b_i \geq \lambda(= CEA_i(c, E))$. When for some $k \in N$ such that $c_k \leq \lambda$, $b_k < c_k$, this is in Case 2-1. Thus, we consider the case where, for each $\ell \in N$ such that $c_\ell \leq \lambda$, $b_\ell = c_\ell(= CEA_\ell(c, E))$. Since for each $i \in N$ such that $c_i > \lambda$, $b_i \geq \lambda(= CEA_i(c, E))$ and for each $\ell \in N$ such that $c_\ell \leq \lambda$, $b_\ell = c_\ell(= CEA_\ell(c, E))$, then for each $m \in N$, $b_m \geq CEA_m(c, E)$. Since $b_{i^{***}} > \lambda(= CEA_{i^{***}}(c, E))$, we have $\sum_{i \in N} b_i > \sum_{i \in N} CEA_i(c, E)$, which contradicts the assumption that $\sum_{i \in N} b_i = \sum_{i \in N} CEA_i(c, E) = E$. Thus, for some $j \in N$ such that $c_j > \lambda$, $b_j < \lambda$. Claimant j can obtain $b_{i^{***}}$ by proposing an appropriate π^j $(\neq \pi^j_\sigma)$. Since $\min\{c_j, b_{i^{***}}\} > \lambda > b_j$, claimant j gains by deviating from σ_j, which contradicts the hypothesis that σ is an SPE of $\Gamma(c, E)$.

By the above discussion, for any $b \neq CEA(c, E)$, $[b, t]$ is not supported as an SPE outcome of $\Gamma(c, E)$.

Next, we show that for any $\tilde{t} \neq 1$, $[CEA(c, E), \tilde{t}]$ is not supported as an SPE outcome of $\Gamma(c, E)$. If $\sum_{i \in N} c_i = E$, then $A(c, E) = \{c\} = \{CEA(c, E)\}$. Therefore, each claimant proposes a pair $(a, \pi) \in A(c, E) \times \Pi$ such that $a = CEA(c, E)$ at period 1. This implies that the game ends at period 1. Thus, in the case $\sum_{i \in N} c_i = E$, for any $\tilde{t} \neq 1$, $[CEA(c, E), \tilde{t}]$ is not supported as an SPE outcome of $\Gamma(c, E)$.

We consider the case $\sum_{i \in N} c_i > E$. Suppose, by contradiction, that there exists an SPE σ' of $\Gamma(c, E)$ whose outcome is $[CEA(c, E), \tilde{t}]$ such that $\tilde{t} \neq 1$. Then, for each $i \in N$, claimant i's payoff is $\delta^{\tilde{t}-1} CEA_i(c, E)$. Let j be the claimant whose claim is minimum among the claims larger than zero. Since $\sum_{i \in N} c_i > E$, such a claimant exists. We show that claimant j can obtain a payoff larger than $\delta^{\tilde{t}-1} CEA_j(c, E)(> 0)$ by deviating from σ'_j.

Let (a^k, π^k) be claimant k's proposal at period 1 under σ', where $k \neq j$. Since under σ', the game ends at period $\tilde{t} \geq 2$, no two claimant propose the same awards vector at period 1, so that $a^k \neq a^j$, where a^j is claimant j's proposal concerning an awards vector at period 1 under σ'. We show that there exists $\ell \in N$ such that $a^k_\ell \geq CEA_j(c, E)$.

Suppose that for each $i \in N$, $a_i^k < CEA_j(c, E)$. By the definitions of CEA and claimant j, for each $i \in N$ such that $c_i > 0$, $CEA_j(c, E) \leq CEA_i(c, E)$. Then, for each $i \in N$ such that $c_i > 0$, $a_i^k < CEA_i(c, E)$. In addition, for each $i \in N$ such that $c_i = 0$, $a_i^k = CEA_i(c, E) = 0$. Therefore, $\sum_{i \in N} a_i^k < \sum_{i \in N} CEA_i(c, E)$, which contradicts the assumption that $\sum_{i \in N} a_i^k = \sum_{i \in N} CEA_i(c, E) = E$. Thus, there exists $\ell \in N$ such that $a_\ell^k \geq CEA_j(c, E)$.

This implies that claimant j can obtain the payoff of $\min\{c_j, a_\ell^k\}$ at period 1 by deviating from σ_j' and proposing a pair consisting of a^k and an appropriate $\pi^j \in \Pi$, because, when claimant j changes his proposal concerning an awards vector a^j into a^k at period 1, a^k becomes the awards vector which receives the highest number of votes. Since $\min\{c_j, a_\ell^k\} \geq CEA_j(c, E) > \delta^{\tilde{t}-1} CEA_j(c, E) > 0$, claimant j gains by deviating from σ_j', which contradicts the hypothesis that σ' is an SPE of $\Gamma(c, E)$. Therefore, $[CEA(c, E), \tilde{t}]$ such that $t \neq 1$ is not supported as an SPE outcome of $\Gamma(c, E)$.

Finally, we show that disagreement is not supported as an SPE outcome. The proof is analogous to the case of $[CEA(c, E), \tilde{t}]$ such that $\tilde{t} \neq 1$. If $\sum_{i \in N} c_i = E$, then the game ends at period 1 by the same reason in the case of $[CEA(c, E), \tilde{t}]$ such that $\tilde{t} \neq 1$. Thus, in the case $\sum_{i \in N} c_i = E$, disagreement is not supported as an SPE outcome of $\Gamma(c, E)$.

We consider the case $\sum_{i \in N} c_i > E$. Suppose, by contradiction, that there exists an SPE σ'' of $\Gamma(c, E)$ whose outcome is disagreement. Then, for each $i \in N$, claimant i's payoff is zero. Let j be the claimant whose claim is larger than zero. Since $\sum_{i \in N} c_i > E$, such a claimant exists. We show that claimant j can obtain a payoff larger than zero by deviating from σ_j''.

Let (a^k, π^k) be claimant k's proposal at period 1 under σ'', where $k \neq j$. Since under σ'', disagreement occurs, then no two claimant propose the same awards vector at period 1, so that $a^k \neq a^j$, where a^j is claimant j's proposal concerning an awards vector at period 1 under σ''.

Since $E > 0$, there exists $\ell \in N$ such that $a_\ell^k > 0$. This implies that claimant j can obtain the payoff of $\min\{c_j, a_\ell^k\}$ at period 1 by deviating from σ_j'' and proposing a pair consisting of a^k and an appropriate $\pi^j \in \Pi$. Since $\min\{c_j, a_\ell^k\} > 0$, claimant j gains by deviating from σ''_j, which contradicts the hypothesis that σ'' is an SPE of $\Gamma(c, E)$. Therefore, disagreement is not supported as an SPE outcome of $\Gamma(c, E)$.

By the above discussions, any SPE outcome of $\Gamma(c, E)$ is $[CEA(c, E), 1]$. $\square$

From Propositions 1 and 2, we have the following main result.

Theorem 1. *For each* $(c, E) \in \mathcal{C}^N$, $[CEA(c, E), 1]$ *is the unique SPE outcome of* $\Gamma(c, E)$.

In the following remarks, we point out interesting relations between the CEA rule and some solution concept of the cooperative game theory. For this theory to be applicable, we need first to define a formal way of associating, with each claims problem, a cooperative problem. Two main classes of such problems have been studied, bargaining problems (Remark 3) and coalitional problems (Remark 4), and accordingly we establish two kinds of relations.

Remark 3. A bargaining problem is a pair (B, d), where B is a subset of $\mathbb{R}^N$ and d is a point of B. The set B is the feasible set consisting of all utility vectors attainable by the group N and d is the disagreement point. A bargaining solution is a function defined on a class of bargaining problems that associates, with each bargaining problem in the class, a unique point in the feasible set of the problem. The **Nash bargaining solution** (Nash, 1950) selects the point maximizing the product of utility gains from d among all points of B dominating d.

Given a claims problem $(c, E) \in \mathcal{C}^N$, its **associated bargaining problem** is the problem with feasible set $B(c, E) = \{a \in \mathbb{R}^N_+ \mid \text{ for each } i \in N, a_i \leq c_i \text{ and } \sum_{i \in N} a_i = E\}$ and disagreement point $d = 0$.

For each $(c, E) \in \mathcal{C}^N$, the outcome chosen by the CEA rule coincides with the outcome chosen by the Nash bargaining solution when applied to $(B(c, E), d)$ (Dagan and Volij, 1993). Therefore, for the bargaining problem associated with a claims problem, our game provides a strategic justification of the Nash bargaining solution.

As a special case of bargaining problem, our results are also applicable to the "Divide-the-Dollar problem," where $(B(c, 1), d)$ such that for each $i \in N$, $c_i \geq 1$.$\Diamond$

Remark 4. A (transferable utility) coalitional problem is a vector $v \equiv (v(S))_{S \subseteq N} \in$

$\mathbb{R}^{2^{|N|}-1}$, where for each coalition $\emptyset \neq S \subseteq N$, $v(S) \in \mathbb{R}$ is the worth of S. A solution is a mapping that associates, with each such problem v, a point in $\mathbb{R}^N$ whose coordinates add up to $v(N)$. The **Dutta-Ray solution** (Dutta and Ray, 1989) selects, for each convex coalitional problem, the payoff vector in the core that is Lorenz-maximal.[18]

Given a claims problem $(c, E) \in \mathcal{C}^N$, its **associated coalitional problem** (O'Neill, 1982; Aumann and Maschler, 1985) is the problem $v(c, E) \in \mathbb{R}^{2^{|N|}-1}$ defined by setting for each $\emptyset \neq S \subseteq N$, $v(c, E)(S) \equiv \max\{E - \sum_{i \in N \setminus S} c_i, 0\}$.

For each $(c, E) \in \mathcal{C}^N$, the outcome chosen by the CEA rule coincides with the outcome chosen by the Dutta-Ray solution when applied to $v(c, E)$ (see, for example, Theorem 14.2 at pp 373 of Thomson, 2019). Therefore, for the coalitional problem associated with a claims problem, our game provides a strategic justification of the Dutta-Ray solution.$\diamondsuit$

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[18]For the definitions of Lorenz-domination and the Dutta-Ray solution, see Dutta and Ray (1989).

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