Squared slack variables in nonlinear symmetric cone programming: Optimality conditions and augmented Lagrangians

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August 12th, 2016

Our setting

Consider the following program.

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) \in \mathcal{K}, \end{aligned} \tag{NSCP}$$

where:

- $f : \mathbb{R}^n \to \mathbb{R}$ is a twice differentiable function,
- $g : \mathbb{R}^n \to \mathcal{E}$ is also a twice differentiable function,
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Objectives

We want to understand optimality conditions for (NSCP).

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In the conic nonlinear programming (NCLP) world...

- $d^\top \nabla^2 L(x, \lambda) d \geq 0$, for $d \in \mathcal{C}$ is **NOT** a necessary condition for local min.
- $d^\top \nabla^2 L(x, \lambda) d > 0$, for $d \in \mathcal{C}, d \neq 0$ is too strong.

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- See also (Forsgreen,2010) and (Jarre, 2012).

Slack variables approach

$$\begin{aligned} \underset{x,y}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) - y \circ y = 0. \end{aligned} \qquad \text{(Slack)}$$

where $\circ$ is a Jordan Product:

1. $y \circ z = y \circ z$,
2. $\langle y \circ z, w \rangle = \langle y, z \circ w \rangle$,
3. $y \circ (y^2 \circ z) = y^2 \circ (y \circ z)$,
4. $\mathcal{K} = \{y \circ y \mid y \in \mathcal{E}\}$

- (Slack) is a run-of-the-mill nonlinear program.
- Optimality conditions for (Slack) are much easier.
- There are many solvers for (Slack) but not many for (NSCP).

Objectives 2

- Examine the difference between optimality conditions and regularity conditions for (NSCP) and (Slack).
- Check the computational prospects of the slack variables approach.

KKT conditions

Let $L(x,\lambda) := f(x) - \langle g(x), \lambda \rangle$.
(x,λ) is a KKT pair for (NSCP) if

$$\nabla_x L(x,\lambda) = \nabla f(x) - \nabla g(x)^* \lambda = 0, \qquad \text{(P1.1)}$$
$$\lambda \in \mathcal{K}, \qquad \text{(P1.2)}$$
$$g(x) \in \mathcal{K}, \qquad \text{(P1.3)}$$
$$\lambda \circ g(x) = 0, \qquad \text{(P1.4)}$$

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$$\lambda \circ g(x) = 0, \tag{P1.4}$$

(x, y, λ) is a KKT triple for (Slack) if:

$$\nabla f(x) - \nabla g(x)^* \lambda = 0, \tag{P2.1}$$
$$\lambda \circ y = 0, \tag{P2.2}$$
$$g(x) - y \circ y = 0. \tag{P2.3}$$

λ is arbitrary.

Regularity Conditions

For (NSCP) we have:

- Mangasarian-Fromovitz: if there exists some d such that

$$g(x) + \nabla g(x)d \in \operatorname{int} \mathcal{K},$$

- Nondegeneracy:

$$\mathcal{E} = T_{\mathcal{K}}(g(x)) + \operatorname{Im} \nabla g(x), \qquad \text{(Nondegeneracy)}$$

- Suppose (x, λ) is KKT for (NSCP) and

$$\operatorname{rank} g(x) + \operatorname{rank} \lambda = m, \qquad (1)$$

  then (x, λ) is said to satisfy the *strict complementarity condition*.

For (Slack) we have the "linear independence constraint qualification" (LICQ).

Second order sufficient condition for (Slack)

The second order sufficient condition for (SOSC-NLP) holds if

$$\langle \nabla_x^2 \mathcal{L}(x, y, \lambda)(v, w), (v, w) \rangle > 0,$$

for every $(v, w) \in \mathbb{R}^n \times \mathcal{E}$ such that $\nabla g(x)v - 2y \circ w = 0$, where $\mathcal{L}$ is the Lagrangian of (Slack).

Proposition

Let $(x, y, \lambda) \in \mathbb{R}^n \times \mathcal{E} \times \mathcal{E}$ be KKT for (Slack)*. The second-order sufficient condition (SOSC-NLP) holds if*

$$\langle \nabla_x^2 L(x, \lambda)v, v \rangle + 2\langle w \circ w, \lambda \rangle > 0 \qquad (2)$$

for every non-zero $(v, w) \in \mathbb{R}^n \times \mathcal{E}$ such that $\nabla g(x)v - 2y \circ w = 0$.

L is the Lagrangian of (NSCP).

Second order necessary condition for (Slack)

Proposition

Let (x, y) *be local min for* (Slack) *and* $(x, y, \lambda) \in \mathbb{R}^n \times \mathcal{E} \times \mathcal{E}$ *be KKT such that LICQ holds. Then the second order necessary condition holds (SONC-NLP):*

$$\langle \nabla_x^2 L(x, \lambda) v, v \rangle + 2\langle w \circ w, \lambda \rangle \geq 0 \quad (3)$$

for every $(v, w) \in \mathbb{R}^n \times \mathcal{E}$ *such that* $\nabla g(x) v - 2y \circ w = 0$.

Relation between KKT points and CQs

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(NSCP)		(Slack)
x satisfies nondegeneracy	$\Longleftrightarrow$	$\exists y$ such that (x, y) satisfies LICQ

Relations between second order sufficient conditions

Suppose $\mathcal{K} = \mathcal{S}^n_+$ or a direct product of second order cones $\mathcal{Q}^n$.

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Proposition (A Sufficient Condition via Slack Variables)

Suppose

1. (x, λ) *is KKT pair for* (NSCP).
2. *Strict complementarity.*
3. *The following inequality is satisfied:*

$$\langle \nabla^2_x L(x, \lambda) v, v \rangle + 2\langle w \circ w, \lambda \rangle > 0 \quad (4)$$

for every non-zero $(v, w) \in \mathbb{R}^n \times \mathcal{E}$ *such that* $\nabla g(x) v - 2\sqrt{g(x)} \circ w = 0$.

Then, x *is a local minimum for* (NSCP).

Relations between second order necessary conditions

Suppose $\mathcal{K} = \mathcal{S}^n_+$ or a direct product of second order cones $\mathcal{Q}^n$.

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(NSCP)		(Slack)
(x, λ) is a KKT pair + strict complementarity + nondegeneracy + SOSC-NSCP	$\Longleftrightarrow$	$\exists y$ such that (x, y, λ) a KKT triple + LICQ + SOSC-NLP

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For any symmetric cone $\mathcal{K}$ we have.

Proposition (A Necessary Condition via Slack Variables)

Suppose x is a local min and

1. (x, λ) *is KKT for* (NSCP)
2. *Strict complementarity.*
3. *Nondegeneracy.*

Then:

$$\langle \nabla^2_x L(x, \lambda)v, v\rangle + 2\langle w \circ w, \lambda\rangle \geq 0 \tag{5}$$

for every (v, w) *such that* $\nabla g(x)v - 2\sqrt{g(x)} \circ w = 0$.

Sharp characterization of positive semidefiniteness

- $\mathcal{S}^m$: $m \times m$ symmetric matrices.
- $y \circ \lambda = \frac{y\lambda + \lambda y}{2}$.

Lemma (L.,Fukuda,Fukushima,2016)

Let $\lambda \in \mathcal{S}^m$. The following statements are equivalent:

i. $\lambda \succeq 0$,

ii. *there exists $y \in \mathcal{S}^m$ such that $y \circ \lambda = 0$ and*

$$\langle w \circ w, \lambda \rangle > 0, \tag{6}$$

for every nonzero $w \in \mathcal{S}^m$ which satisfies $y \circ w = 0$.

For any y satisfying (6) we have $\operatorname{rank} \lambda = m - \operatorname{rank} y$. Moreover, if σ and σ' are nonzero eigenvalues of y, then $\sigma + \sigma' \neq 0$.

Extension to symmetric cones

Suppose $\mathcal{E}$ is a Jordan algebra and $\mathcal{K} = \mathcal{K}^1 \times \ldots \times \mathcal{K}^s$.

Lemma

Let $\lambda \in \mathcal{E}$. The following statements are equivalent:

i. $\lambda \in \mathcal{K}$,

ii. *there exists $y \in \mathcal{E}$ such that*

$$y \circ \lambda = 0 \text{ and } \langle w \circ w, \lambda \rangle > 0, \tag{7}$$

for every nonzero $w \in \mathcal{E}$ which satisfies $y \circ w = 0$.

Suppose y satisfies Any y satisfying (7) we have that $\operatorname{rank} \lambda = m - \operatorname{rank} y$. *Moreover, if σ and σ' are nonzero eigenvalues of* **the same block** *of y, then $\sigma + \sigma' \neq 0$.*

Is it feasible to use (Slack)?

- There are previous results for linear SDPs by Burer and Monteiro in 2003.
- we used PENLAB to solve the same problems via both (NSCP) and (Slack).

Hock-Schittkowski problem 71

$$
\begin{array}{ll}
\underset{x\in\mathbb{R}^6}{\text{minimize}} & x_1x_4(x_1+x_2+x_3)+x_3 \\
\text{subject to} & x_1x_2x_3x_4 - x_5 - 25 = 0, \\
& x_1^2+x_2^2+x_3^2+x_4^4 - x_6 - 40 = 0, \\
& \begin{pmatrix} x_1 & x_2 & 0 & 0 \\ x_2 & x_4 & x_2+x_3 & 0 \\ 0 & x_2+x_3 & x_4 & x_3 \\ 0 & 0 & x_3 & x_1 \end{pmatrix} \in \mathcal{S}^4_+, \\
& 1 \le x_i \le 5,\ i=1,2,3,4; \quad x_i \ge 0,\ i=5,6.
\end{array} \tag{HS}
$$

Table: Slack vs "native" for (HS)

	functions	gradients	Hessians	iterations	time (s)	opt. value
slack	110	57	44	13	0.54	87.7105
native	123	71	58	13	0.57	87.7105

The closest correlation matrix problem - simple version

Let H be a $m \times m$ symmetric matrix H with diagonal entries equal to one.

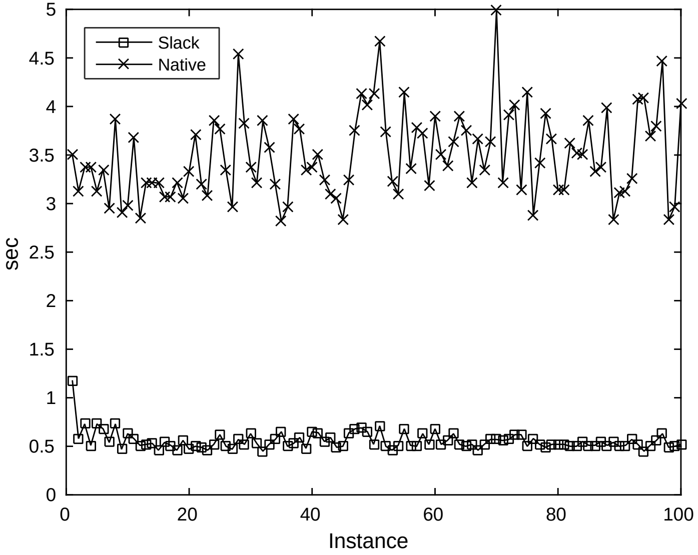
$$\begin{array}{ll} \underset{x}{\text{minimize}} & \langle X - H, X - H \rangle \\ \text{subject to} & X_{ii} = 1 \qquad \forall i, \\ & X \in \mathcal{K}. \end{array} \tag{Cor}$$

Generated 100 symmetric matrices H such that the diagonal elements are all 1 and other elements are uniform random numbers between -1 and 1.

Table: Comparison between native and slack

	Slack			Native		
m	mean (s)	min (s)	max (s)	mean (s)	min (s)	max (s)
5	0.090	0.060	0.140	0.201	0.130	0.250
10	0.153	0.120	0.230	0.423	0.330	0.630
15	0.287	0.210	0.430	1.306	1.020	1.950
20	0.556	0.450	1.180	3.491	2.820	4.990

Case-by-case comparison (m = 20)



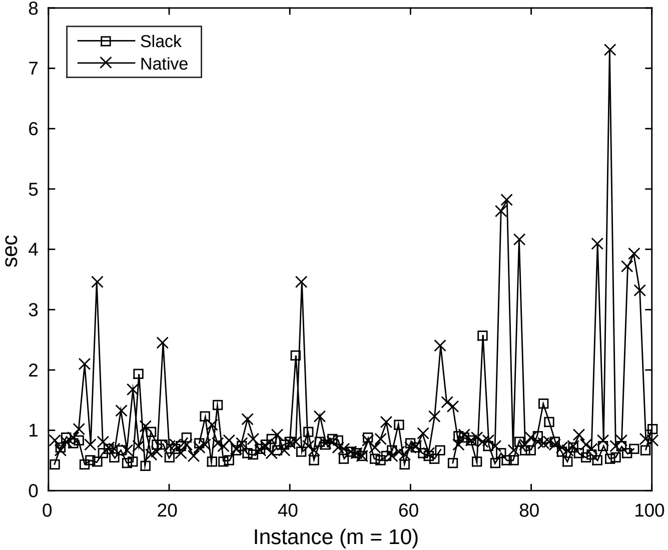
The closest correlation matrix problem - extended version

$$\begin{aligned} \underset{X,z}{\text{minimize}} \quad & \langle zX - H, zX - H \rangle \\ \text{subject to} \quad & zX_{ii} = 1 \qquad \forall i, \\ & I_m \preceq X \preceq \kappa I_m, \end{aligned} \tag{Cor-Ext}$$

Table: Comparison between Native and Slack

	Slack				Native			
m	mean (s)	min (s)	max (s)	fail	mean (s)	min (s)	max (s)	fail
5	0.236	0.130	0.830	15	0.445	0.250	2.130	1
10	0.741	0.420	2.580	3	1.206	0.580	7.300	0
15	4.651	2.090	26.96	15	3.809	1.960	14.12	0
20	24.32	15.20	69.34	8	9.288	5.150	36.81	0

Case-by-case comparison (m = 10)



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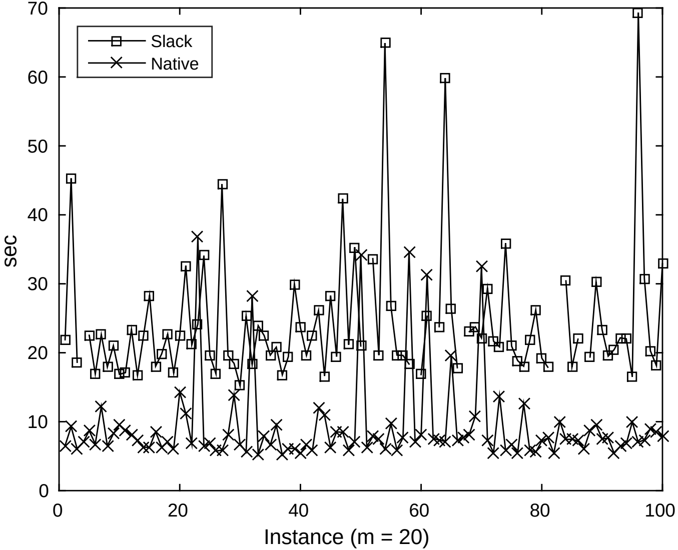

We have the augmented Lagrangian

$$\mathfrak{L}_\rho^{\mathsf{Slack}}(x, y, \lambda) = f(x) - \langle g(x) - y \circ y, \lambda \rangle + \frac{\rho}{2}\|g(x) - y \circ y\|^2. \tag{AL-SLACK}$$

$$\min_y \mathfrak{L}_\rho^{\mathsf{Slack}}(x, y, \lambda) = f(x) + \frac{1}{2\rho}\left(-\|\lambda\|^2 + \||\lambda - \rho g(x)|_+\|^2\right),$$

where $|x|_+$ is the orthogonal projection of x on $\mathcal{K}$. This suggests this:

$$\mathfrak{L}_\rho^{\mathsf{Sym}}(x, \lambda) = f(x) + \frac{1}{2\rho}\left(-\|\lambda\|^2 + \||\lambda - \rho g(x)|_+\|^2\right) \tag{AL-CONE}$$

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$$\min_y \mathfrak{L}_\rho^{\text{Slack}}(x, y, \lambda) = f(x) + \frac{1}{2\rho}\left(-\|\lambda\|^2 + \||\lambda - \rho g(x)|_+\|^2\right),$$

We have the augmented Lagrangian

$$\mathfrak{L}_{\rho}^{\mathrm{Slack}}(x, y, \lambda) = f(x) - \langle g(x) - y \circ y, \lambda \rangle + \frac{\rho}{2} \| g(x) - y \circ y \|^2. \tag{AL-SLACK}$$

$$\min_{y} \mathfrak{L}_{\rho}^{\mathrm{Slack}}(x, y, \lambda) = f(x) + \frac{1}{2\rho} \left(-\|\lambda\|^2 + \| |\lambda - \rho g(x)|_+ \|^2 \right),$$

where $|x|_+$ is the orthogonal projection of x on $\mathcal{K}$. This suggests this:

$$\mathfrak{L}_{\rho}^{\mathrm{Sym}}(x, \lambda) = f(x) + \frac{1}{2\rho} \left(-\|\lambda\|^2 + \| |\lambda - \rho g(x)|_+ \|^2 \right) \tag{AL-CONE}$$

Augmented Lagrangian Method for (Slack)

1. Choose initial points x_1, y_1, initial multipliers (μ_1, λ_1) and an initial penalty ρ_1.
2. $k \leftarrow 1$.
3. $(x_{k+1}, y_{k+1}) \leftarrow \mathrm{argmin}_{x,y}\, \mathfrak{L}^{\mathsf{Slack}}_{\rho_k}(\cdot, \cdot, \lambda_k)$.
4. $\lambda_{k+1} \leftarrow \lambda_k - \rho_k(g(x_{k+1}) - y_{k+1} \circ y_{k+1})$.
5. Choose a new penalty parameter ρ_{k+1} with $\rho_{k+1} \geq \rho_{k+1}$.
6. Let $k \leftarrow k + 1$ and return to Step 3.

Augmented Lagrangian Method for (NSCP)

1 Choose an initial point x_1, initial multipliers (μ_1, λ_1) and an initial penalty ρ_1.
2 $k \leftarrow 1$.
3 $x_{k+1} \leftarrow \operatorname{argmin}_x \mathfrak{L}^{\mathsf{Sym}}_{\rho_k}(\cdot, \lambda_k)$.
4 $\mu_{k+1} \leftarrow \mu_k - \rho_k h(x_{k+1})$.
5 $\lambda_{k+1} \leftarrow |\lambda_k - \rho_k g(x_{k+1})|_+$.
6 Choose a new penalty parameter ρ_{k+1} with $\rho_{k+1} \geq \rho_{k+1}$.
7 Let $k \leftarrow k + 1$ and return to Step 3.

- See (Bertsekas, 1982) for the case $\mathcal{K} = \mathbb{R}^m_+$
- For the case $\mathcal{K} = \mathcal{S}^m_+$, see *"The rate of convergence of the augmented Lagrangian method for nonlinear semidefinite programming"* by Defeng Sun, Jie Sun and Liwei Zhang.

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See

- B. F. L., Ellen H. Fukuda and Masao Fukushima. *Optimality conditions for nonlinear semidefinite programming via squared slack variables*. `arxiv:1512.05507` To appear in Math Prog, 2016
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Thank you!