Polynomial Time
Iterative Methods
for
Integer Programming

Shmuel Onn

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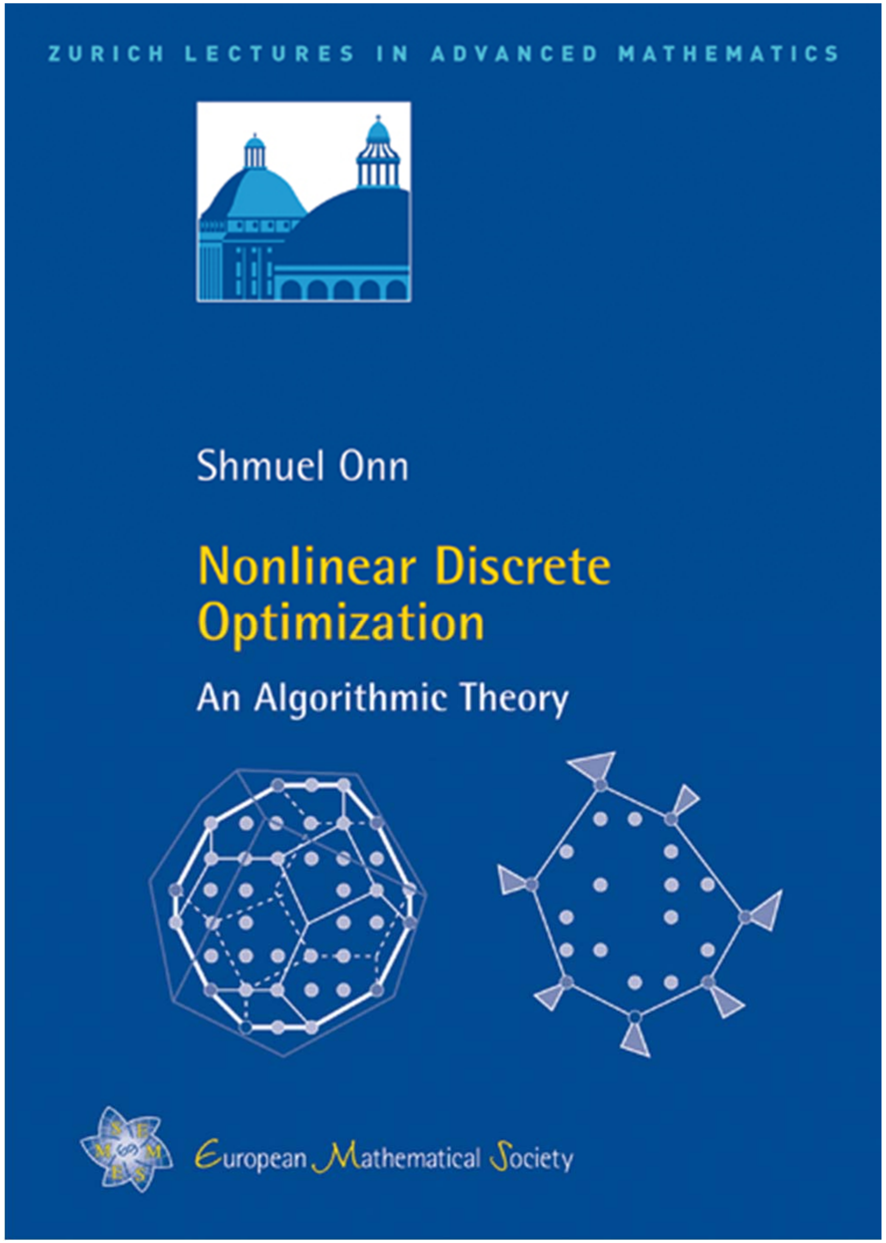

Background in my book:

Theory of Graver bases for integer programming

Available electronically from my homepage

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(Non)-Linear Integer Programming

The problem is: $\min/\max\{\, f(x) \;:\; Ax \le b,\;\; l \le x \le u,\;\; x \in Z^n \,\}$

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(Non)-Linear Integer Programming

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with data: A: integer $m \times n$ matrix

b: right-hand side in Z^m

l,u: lower/upper bounds in Z^n

f: function from Z^n to R

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(Non)-Linear Integer Programming

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Our theory enables polynomial time solution of broad natural universal (non)-linear integer programs in variable dimension

(with De Loera, Hemmecke, Lee, Romanchuk, Rothblum, Weismantel)

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Graver Bases

and

Nonlinear Integer Programming

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x is conformal-minimal if no other y in same orthant has all $|y_i| \le |x_i|$

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non-circuits: $\pm(1\ {-1}\ 1)$

Some Theorems on
(Non)-Linear Integer Programming

Theorem 1: **separable convex minimization** in **polytime** with **$G(A)$**:

$$\min \{\sum f_i(x_i) \ : \ Ax = b, \ \ l \leq x \leq u, \ \ x \in Z^n\}$$

Reference: A polynomial oracle-time algorithm for convex integer minimization (Hemmecke, Onn, Weismantel) Mathematical Programming, 2011

Some Theorems on
(Non)-Linear Integer Programming

Theorem 2: **quadratic minimization** in **polytime** with **G(A)**:

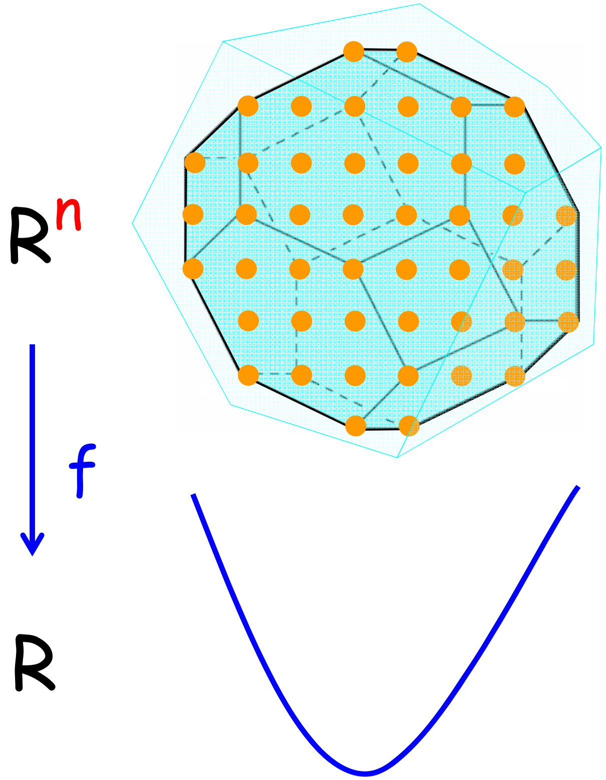
$$\min \{x^T V x \; : \; Ax = b, \;\; l \le x \le u, \;\; x \in Z^n\}$$

where V lies in cone $K_2(A)$ of possibly indefinite matrices, enabling minimization of some convex and some non-convex quadratics.

Reference: The quadratic Graver cone, quadratic integer minimization & extensions (Lee, Onn, Romanchuk, Weismantel), Mathematical Programming, 2012

The Main Iterative Algorithm

Proof of Theorem 1

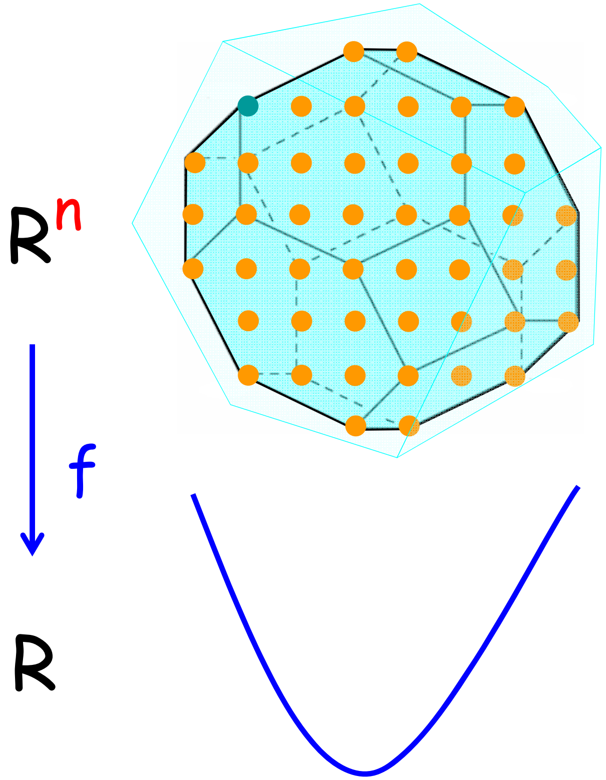


To solve $\min\{\sum f_i(x_i) : Ax = b,\ l \le x \le u,\ x \in Z^n\}$

with the Graver basis $G(A)$

Do:

Proof of Theorem 1



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1. Find initial point by auxiliary program

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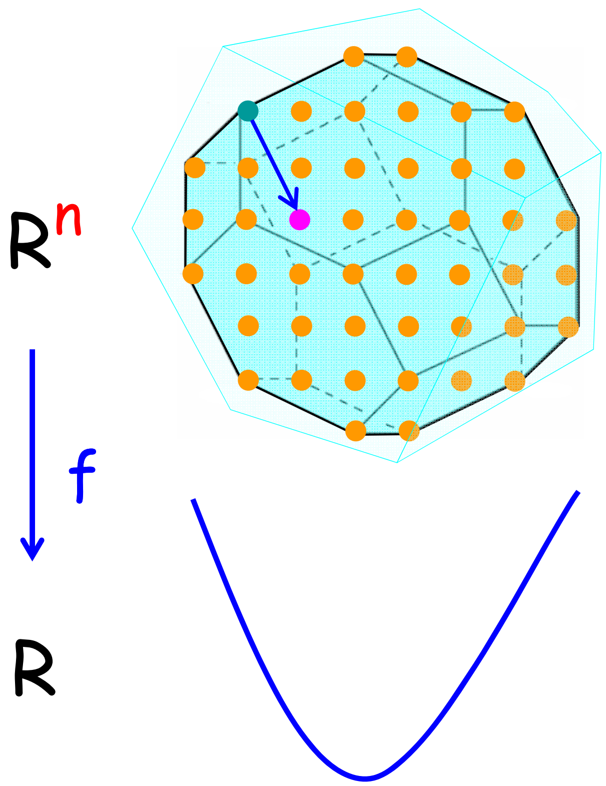

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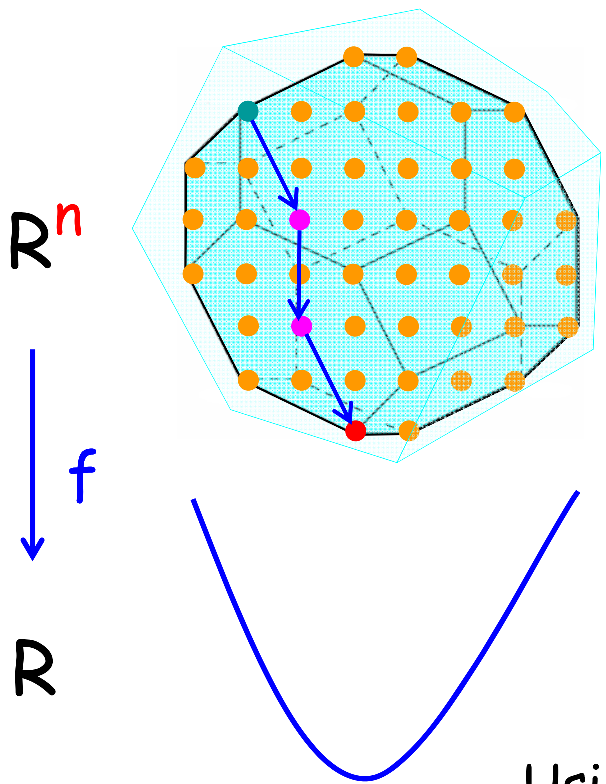

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Using supermodality of f and integer Caratheodory theorem (Cook-Fonlupt-Schrijver, Sebo) we can show polytime convergence to some optimal solution

N-Fold Integer Programming

N-Fold Products

The n-fold product of an $(r,s) \times t$ bimatrix $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ is the $(r+ns) \times nt$ matrix

$$A^{(n)} = \underbrace{\begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_2 \end{pmatrix}}_{n} .$$

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Lemma: For fixed A we can compute the Graver basis $G(A^{(n)})$ in polynomial time $O(n^{g(A)})$ with $g(A)$ the Graver complexity of A.

(Non)-Linear N-Fold Integer Programming

Theorem: for various f can solve in polynomial time $O(n^{g(A)} L)$:

$$\min\{f(x) : A^{(n)}x = b,\ l \le x \le u,\ x \in Z^{nt}\}$$

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References: see Nonlinear Discrete Optimization (Onn), Zurich Lectures in Advanced Mathematics, European Mathematical Society, 2010

N-Fold Integer Programming

is

Fixed-Parameter Tractable

N-Fold IP is Fixed-Parameter Tractable

Reference: N-fold integer programming in cubic time
(Hemmecke, Onn, Romanchuk) Mathematical Programming, 2013

N-Fold IP is Fixed-Parameter Tractable

Theorem: For any fixed bimatrix A, the following linear n-fold integer program is solvable in fixed-parameter time $O(n^3 L)$:

$$\max\{wx : A^{(n)}x = b,\ l \le x \le u,\ x \in Z^{nt}\}$$

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Proof very rough idea: in the iterative algorithm, at each iteration, can find a Graver-best step without computing the entire Graver basis.

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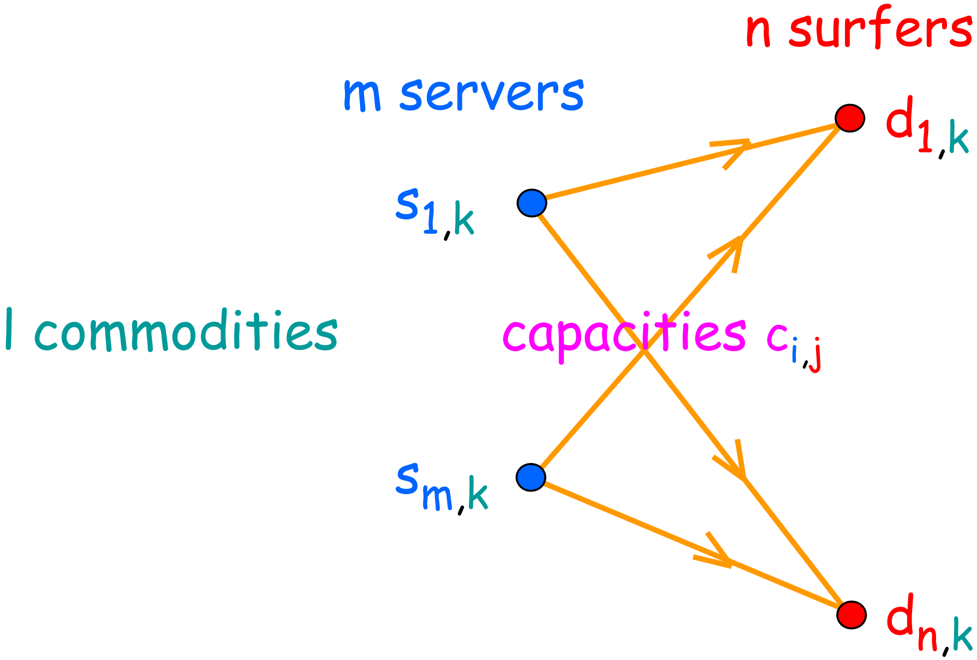
An Application:

Multicommodity Flows

Multicommodity Flows

Find flow of l commodities from m servers to n surfers satisfying given supplies $s_{i,k}$, demands $d_{j,k}$ and capacities $c_{i,j}$ of total bit size L



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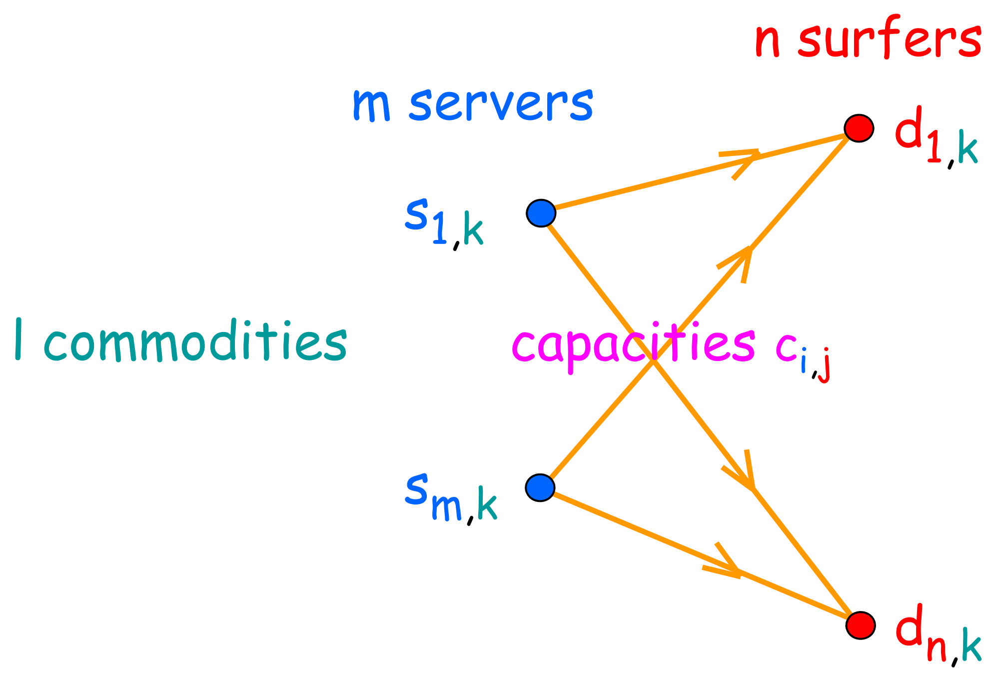

With l=2 or m=3 it is NP-complete so assume both l,m are parameters

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Multicommodity Flows

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2008: polynomial time $O(n^{g(l,m)} L)$ with Graver complexity $g(l,m)$ exponential in l,m (De Loera, Hemmecke, Onn, Weismantel) (theory of n-fold IP)

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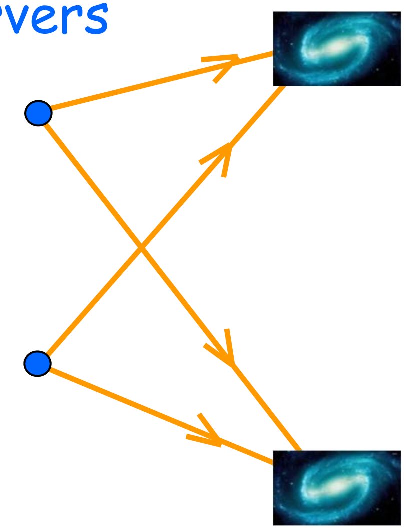
Open: algorithm that is both fixed-parameter tractable and strongly polynomial ?

Multicommodity Flows

Huge version: surfers come in huge clouds of t types



t surfer types

$n_1 + \ldots + n_t = n$

binary encoded

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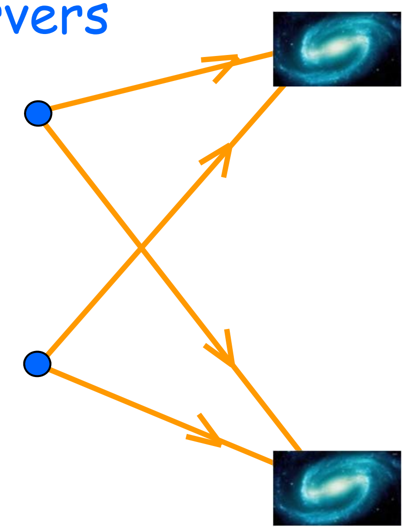

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2016 (Onn): fixed-parameter tractable with parameters l,t, variable m, huge n

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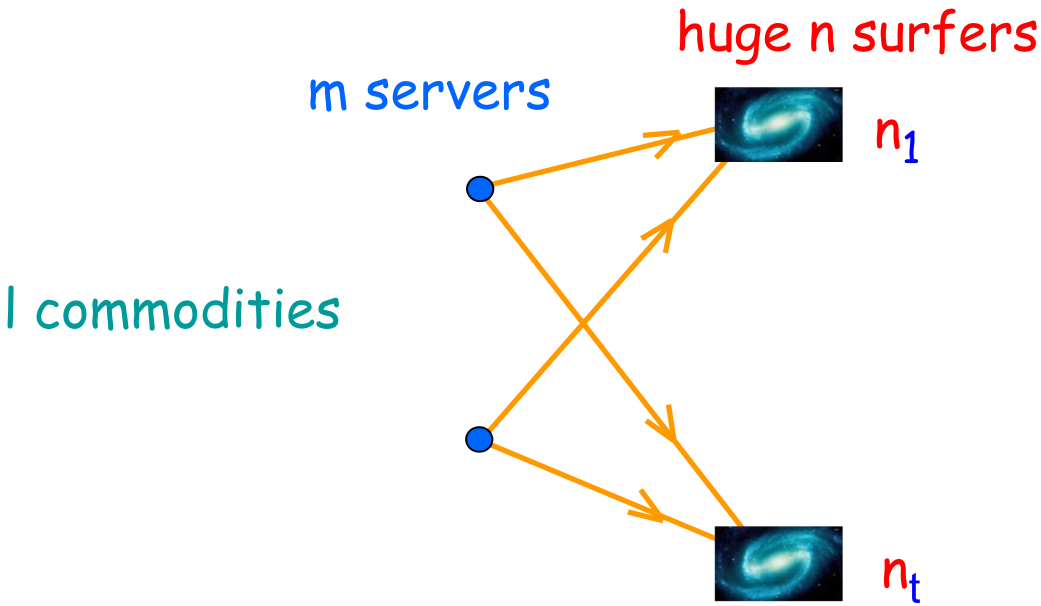

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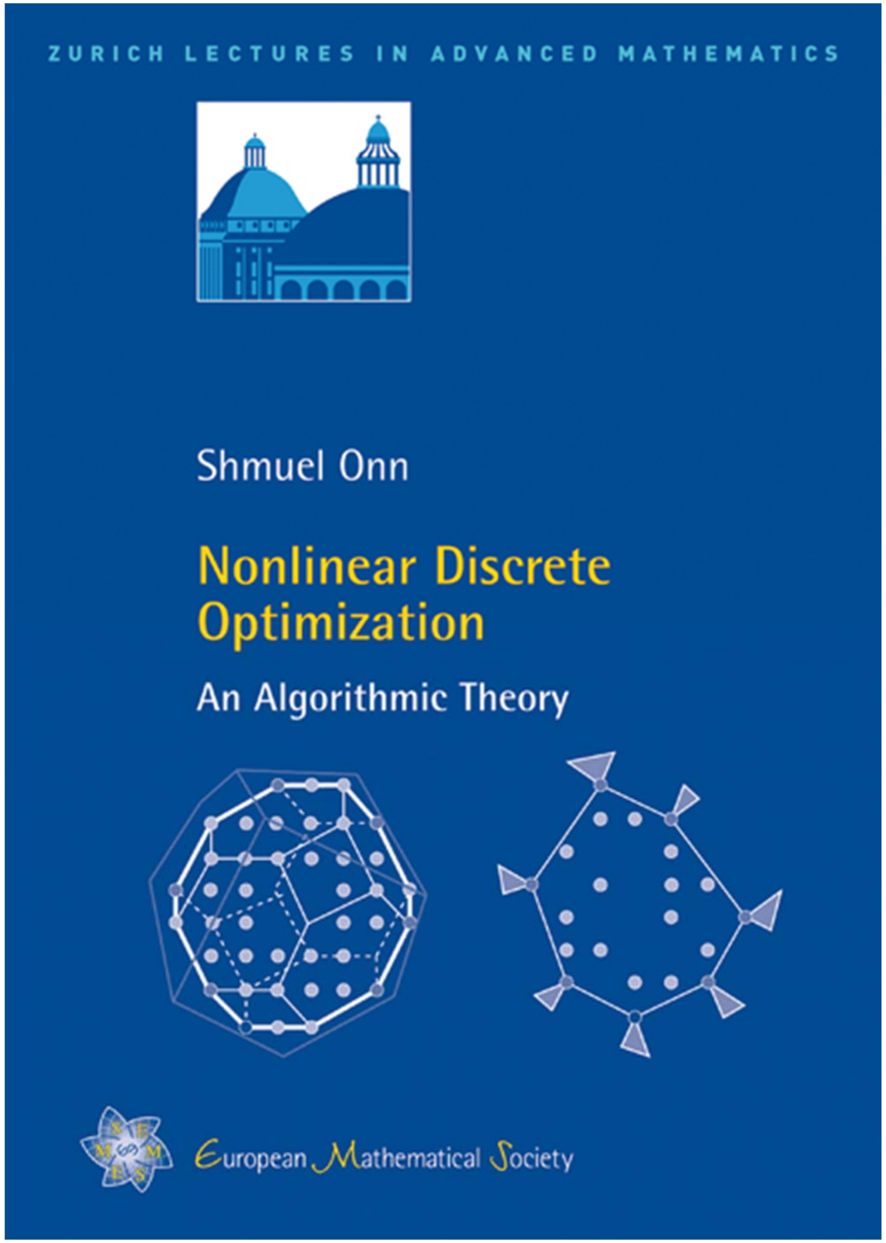
Open: 4-dimensional huge tables are only known to be in NP intersect coNP

Some Bibliography

(available at http://ie.technion.ac.il/~onn)

- The complexity of 3-way tables (SIAM J. Comp.)
- Convex combinatorial optimization (Disc. Comp. Geom.)
- Markov bases of 3-way tables (J. Symb. Comp.)
- All linear and integer programs are slim 3-way programs (SIAM J. Opt.)
- Graver complexity of integer programming (Annals Combin.)
- N-fold integer programming (Disc. Opt. in memory of Dantzig)
- Convex integer maximization via Graver bases (J. Pure App. Algebra)
- Polynomial oracle-time convex integer minimization (Math. Prog.)
- The quadratic Graver cone, quadratic integer minimization & extensions (Math Prog.)
- **N-fold integer programming in cubic time** (Math. Prog.)
- **Huge tables are fixed-parameter tractable via unimodular integer Caratheodory**



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