Mixed-Integer SOCP in optimal contribution selection of tree breeding

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Agenda

1. Optimal Contribution Problem
2. Equally Contribution Problem
3. Conic (LP, SOCP, SDP) Relaxations
4. Steep-Ascent Method
5. Numerical Results

$$\begin{aligned} \max \quad : \quad & \boldsymbol{g}^T \boldsymbol{x} \\ \text{s.t.} \quad : \quad & \boldsymbol{e}^T \boldsymbol{x} = 1 \\ & ||\boldsymbol{B}\boldsymbol{x}|| \leq \sqrt{2\theta} \\ & \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u} \\ & x_1, \ldots, x_Z \in \left\{0, \frac{1}{N}\right\} \end{aligned}$$

SOCP

Integer

Optimal Contribution Problem

pine orchard

contributions

x_i 37% 42% 5% 16%

genotype candidates

price

g_i \$160 \$180 \$20 \$50

$$\max_{\boldsymbol{x}\in\mathbb{R}^z} \; : \; \boldsymbol{g}^T\boldsymbol{x} \qquad \text{best performance}$$

$$\text{s.t.} \; : \; \boldsymbol{e}^T\boldsymbol{x} = 1 \qquad \text{unity}$$

$$\boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u} \qquad \text{bounds}$$

$$\frac{\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}}{2} \leq \theta \qquad \text{genetic diversity}$$

QCP (quadratic-constrained problem)

Existing methods for Optimal Contribution Problems

Meuwissen (1997) : Lagrangian multiplier method
implemented in GENCONT
(www.genebankdata.cgn.wur.nl/gencont/gencont.html)

$$\max_{\boldsymbol{x}\in\mathbb{R}^Z} : \boldsymbol{g}^T\boldsymbol{x}$$
$$\text{s.t.} : \boldsymbol{e}^T\boldsymbol{x} = 1$$
$$\sout{\boldsymbol{l} \le \boldsymbol{x} \le \boldsymbol{u}}$$
$$\frac{\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}}{2} \le \theta \;\Rightarrow\; \frac{\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}}{2} = \theta$$
$$\text{if } x_i < l_i, \text{ fix } x_i = l_i$$

Yamashita et al. (2015) : SOCP Approach

$$\max_{\boldsymbol{x}\in\mathbb{R}^Z} : \boldsymbol{g}^T\boldsymbol{x}$$
$$\text{s.t.} : \boldsymbol{e}^T\boldsymbol{x} = 1$$
$$\boldsymbol{l} \le \boldsymbol{x} \le \boldsymbol{u}$$
$$||\boldsymbol{B}\boldsymbol{x}|| \le \sqrt{2\theta}$$

$\boldsymbol{A} = \boldsymbol{B}^T\boldsymbol{B}$ $(\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \le 2\theta \Leftrightarrow ||\boldsymbol{B}\boldsymbol{x}|| \le \sqrt{2\theta})$

$\boldsymbol{A}^{-1}$ is sparse and has special structure

Pong-Wong and Wooliams (2007) : SDP approach
implemented in TREEPLAN

$$\max_{\boldsymbol{x}\in\mathbb{R}^Z} : \boldsymbol{g}^T\boldsymbol{x}$$
$$\text{s.t.} : \boldsymbol{e}^T\boldsymbol{x} = 1$$
$$\boldsymbol{l} \le \boldsymbol{x} \le \boldsymbol{u}$$
$$\frac{\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}}{2} \le \theta \;\Rightarrow\; \begin{pmatrix} 2\theta & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{A}^{-1} \end{pmatrix} \text{ is positive semide}$$

$\boldsymbol{A}$ is positive definite

Computation time (in second)

Z	GENCONT	SDP	SOCP
2,045	67.43	70.21	0.09
15,100	OOM	21994.87	2.92
100,100	OOM	OOM	17.92

OOM is out of memory

# Unequally Contribution Problems and Equally Contribution Problems

Unequally

$$\max_{\boldsymbol{x}\in\mathbb{R}^Z} \; : \; \boldsymbol{g}^T\boldsymbol{x}$$
$$\text{s.t.} \; : \; \boldsymbol{e}^T\boldsymbol{x} = 1$$
$$\boldsymbol{l} \le \boldsymbol{x} \le \boldsymbol{u}$$
$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \le 2\theta$$

Equally

$$\max_{\boldsymbol{x}\in\mathbb{R}^Z} \; : \; \boldsymbol{g}^T\boldsymbol{x}$$
$$\text{s.t.} \; : \; \boldsymbol{e}^T\boldsymbol{x} = 1$$
$$\cancel{\boldsymbol{l} \le \boldsymbol{x} \le \boldsymbol{u}}$$
$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \le 2\theta$$
$$x_1, x_2, \ldots, x_Z \in \left\{0, \frac{1}{N}\right\}$$

Choose exatctly N genotypes from Z candidates.

$Z = 4, N = 2$

contributions x_i: 0%, 50%, 50%, 0%

Genotype candidates

price g_i: \$160, \$180, \$20, \$50

Existing methods for *Equally* Contribution Problems

Meuwissen (1997) : Lagrangian multiplier method
implemented in GENCONT
(www.genebankdata.cgn.wur.nl/gencont/gencont.html)

$$\max_{\boldsymbol{x}\in\mathbb{R}^Z} \; : \; \boldsymbol{g}^T\boldsymbol{x}$$

$$\text{s.t.} \; : \; \boldsymbol{e}^T\boldsymbol{x} = 1$$

~~$\boldsymbol{0} \le \boldsymbol{x} \le \boldsymbol{e}/N$~~

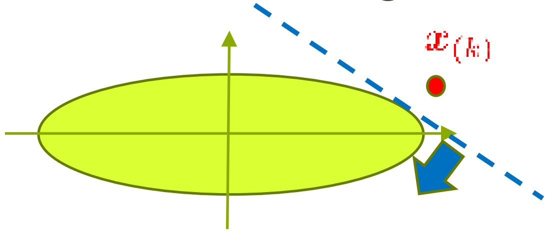
$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = 2\theta$$

$$\text{if } x_i < 0, \text{ fix } x_i = 0$$

Mullin and Belotti (2016) : Branch-and-bound
& outer approximation
implemented in OPSEL (www.skogforsk.se/opsel)

$$\max_{\boldsymbol{x}\in\mathbb{R}^Z} \; : \; \boldsymbol{g}^T\boldsymbol{x}$$

$$\begin{aligned} \text{s.t.} \; : \; & \boldsymbol{e}^T\boldsymbol{x} = 1 \\ & \boldsymbol{l} \le \boldsymbol{x} \le \boldsymbol{u} \\ & \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \le 2\theta \\ & x_i \in \left\{0, \frac{1}{N}\right\} \end{aligned}$$



When $\boldsymbol{x}_{(k)}{}^T\boldsymbol{A}\boldsymbol{x}_{(k)} > 2\theta$,
add $\boldsymbol{x}_{(k)}{}^T\boldsymbol{A}\boldsymbol{x} \le \sqrt{2\theta\; \boldsymbol{x}_{(k)}{}^T\boldsymbol{A}\boldsymbol{x}_{(k)}}$

CPLEX can handle mixed-integer SOCP

$$\max_{\boldsymbol{x}\in\mathbb{R}^Z} \; : \; \boldsymbol{g}^T\boldsymbol{x}$$

$$\begin{aligned} \text{s.t.} \; : \; & \boldsymbol{e}^T\boldsymbol{x} = 1 \\ & \boldsymbol{l} \le \boldsymbol{x} \le \boldsymbol{u} \\ & \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \le 2\theta \\ & x_i \in \left\{0, \frac{1}{N}\right\} \end{aligned}$$

$Z = 200, N = 50$ (Choose 50 genotypes from 200 candidates)

Solver	time	duality gap
OPSEL	12 hours	0.5%
CPLEX	1 day	11.45%
CPLEX	1 month	11.43%

Finding feasible solutions is very hard.

We should seek an approximate solution in a practical time.

# SDP relaxation for Equally Contribution Problem

(1)
$$\begin{aligned} \max_{\boldsymbol{x}\in\mathbb{R}^Z} &: \boldsymbol{g}^T\boldsymbol{x} \\ \text{s.t.} &: \boldsymbol{e}^T\boldsymbol{x} = 1 \\ &\quad \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \le 2\theta \\ &\quad x_i \in \left\{0, \frac{1}{N}\right\} \end{aligned}$$

$$y_i = 2Nx_i - 1 \Leftrightarrow x_i = \frac{y_i+1}{2N}$$

(2)
$$\begin{aligned} \max_{\boldsymbol{y}\in\mathbb{R}^Z} &: \frac{1}{2N}\boldsymbol{g}^T\boldsymbol{y} + \frac{1}{2N}\boldsymbol{g}^T\boldsymbol{e} \\ \text{s.t.} &: \boldsymbol{e}^T\boldsymbol{y} = 2N - Z \\ &\quad (\boldsymbol{e}\boldsymbol{e}^T)\bullet(\boldsymbol{y}\boldsymbol{y}^T) = (2N-Z)^2 \\ &\quad \boldsymbol{y}^T\boldsymbol{A}\boldsymbol{y} \le 2\theta N^2 \\ &\quad y_i \in \{-1, 1\} \end{aligned}$$

$$\boldsymbol{A}\bullet\boldsymbol{B} = \sum_{i=1}^n\sum_{j=1}^n A_{ij}B_{ij}$$

$$\hat{\boldsymbol{g}} := \frac{1}{4N}\boldsymbol{g},\ \bar{g} = \frac{1}{4N}\boldsymbol{g}^T\boldsymbol{e},\ \bar{N} = 2N - Z$$

(3)
$$\begin{aligned} \max &: 2\hat{\boldsymbol{g}}^T\boldsymbol{y} + \bar{g} \\ \text{s.t.} &: \boldsymbol{e}^T\boldsymbol{y} = \bar{N} \\ &\quad (\boldsymbol{e}\boldsymbol{e}^T)\bullet\boldsymbol{Y} = \bar{N}^2 \\ &\quad \boldsymbol{A}\bullet\boldsymbol{Y} \le 2\theta N^2 \\ &\quad \boldsymbol{Y} = \boldsymbol{y}\boldsymbol{y}^T \\ &\quad y_i \in \{-1, 1\} \end{aligned}$$

$$\boldsymbol{Y} = \boldsymbol{y}\boldsymbol{y}^T$$
$$\Updownarrow$$
$$\begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} \succeq \boldsymbol{O},\ \text{rank}\begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} = 1$$
$\Downarrow$ relaxation
$$\begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} \succeq \boldsymbol{O}.$$

(4)
$$\begin{aligned} \max &: \begin{pmatrix} 0 & \hat{\boldsymbol{g}}^T \\ \hat{\boldsymbol{g}} & \boldsymbol{O} \end{pmatrix}\bullet\begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} + \bar{g} \\ \text{s.t.} &: \begin{pmatrix} 0 & \boldsymbol{e}^T \\ \boldsymbol{e} & \boldsymbol{O} \end{pmatrix}\bullet\begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} = \bar{N} \\ &\quad \begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{e}\boldsymbol{e}^T \end{pmatrix}\bullet\begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} = \bar{N}^2 \\ &\quad \begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{A} \end{pmatrix}\bullet\begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} \le 2\theta N^2 \\ &\quad Y_{ii} = 1 \\ &\quad \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} \succeq \boldsymbol{O}. \end{aligned}$$

SDP relaxation problem

# LP relaxation for Equally Contribution Problem

SDP relaxation problem

$$
\begin{aligned}
\max \quad &: \begin{pmatrix} 0 & \hat{\boldsymbol{g}}^T \\ \hat{\boldsymbol{g}} & \boldsymbol{O} \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} + \bar{g} \\
\text{s.t.} \quad &: \begin{pmatrix} 0 & \boldsymbol{e}^T \\ \boldsymbol{e} & \boldsymbol{O} \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} = \bar{N} \\
& \begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{e}\boldsymbol{e}^T \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} = \bar{N}^2 \\
& \begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{A} \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} \le 2\theta N^2 \\
& Y_{ii} = 1 \\
& \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} \succeq \boldsymbol{O}.
\end{aligned}
$$

$$
\begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} \succeq \boldsymbol{O}
$$

$$
\overset{\color{red}{Relaxation}}{\Rightarrow} \begin{pmatrix} 1 & y_i \\ y_i & Y_{ii} \end{pmatrix} \succeq \boldsymbol{O}, \begin{pmatrix} Y_{ii} & Y_{ij} \\ Y_{ij} & Y_{jj} \end{pmatrix} \succeq \boldsymbol{O}
$$

$$
\Leftrightarrow Y_{ii} \ge y_i^2, Y_{ii}Y_{jj} \ge Y_{ij}^2
$$

$$
\overset{Y_{ii}=1}{\Leftrightarrow} 1 \ge y_i^2, 1 \ge Y_{ij}^2
$$

$$
\Leftrightarrow -1 \le y_i \le 1, -1 \le Y_{ij} \le 1
$$

LP relaxation problem

$$
\begin{aligned}
\max \quad &: \begin{pmatrix} 0 & \hat{\boldsymbol{g}}^T \\ \hat{\boldsymbol{g}} & \boldsymbol{O} \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} + \bar{g} \\
\text{s.t.} \quad &: \begin{pmatrix} 0 & \boldsymbol{e}^T \\ \boldsymbol{e} & \boldsymbol{O} \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} = \bar{N} \\
& \begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{e}\boldsymbol{e}^T \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} = \bar{N}^2 \\
& \begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{A} \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} \le 2\theta N^2 \\
& Y_{ii} = 1 \\
& -1 \le y_i \le 1 \\
& -1 \le Y_{ij} \le 1 \ (i \ne j).
\end{aligned}
$$

# SOCP relaxation for Equally Contribution Problem

Equally Contribution Problem

$$\begin{aligned} \max_{\boldsymbol{x}\in\mathbb{R}^Z} \quad &: \quad \boldsymbol{g}^T\boldsymbol{x} \\ \text{s.t.} \quad &: \quad \boldsymbol{e}^T\boldsymbol{x} = 1 \\ & \quad\ ||\boldsymbol{B}\boldsymbol{x}|| \leq \sqrt{2\theta} \\ & \quad\ x_i \in \left\{0, \frac{1}{N}\right\} \end{aligned}$$

$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \leq 2\theta \Leftrightarrow ||\boldsymbol{B}\boldsymbol{x}|| \leq \sqrt{2\theta} \ (\text{by } \boldsymbol{A} = \boldsymbol{B}^T\boldsymbol{B})$$

SOCP relaxation

$$\begin{aligned} \max_{\boldsymbol{x}\in\mathbb{R}^Z} \quad &: \quad \boldsymbol{g}^T\boldsymbol{x} \\ \text{s.t.} \quad &: \quad \boldsymbol{e}^T\boldsymbol{x} = 1 \\ & \quad\ ||\boldsymbol{B}\boldsymbol{x}|| \leq \sqrt{2\theta} \\ & \quad\ 0 \leq x_i \leq \frac{1}{N} \end{aligned}$$

This problem is a piece of cake,
due to the structure of B.

The relation between three conic relaxations

$$OPT_{ECP} = \max_{\boldsymbol{x}\in\mathbb{R}^Z} \; : \; \boldsymbol{g}^T\boldsymbol{x}$$
$$\text{s.t.} \; : \; \boldsymbol{e}^T\boldsymbol{x} = 1, \quad ||\boldsymbol{B}\boldsymbol{x}|| \le \sqrt{2\theta}, \quad x_i \in \left\{0, \frac{1}{N}\right\}$$

$$OPT_{SDP} = \max \; : \; \begin{pmatrix} 0 & \hat{\boldsymbol{g}}^T \\ \hat{\boldsymbol{g}} & \boldsymbol{O} \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} + \bar{g}$$
$$\text{s.t.} \; : \; \begin{pmatrix} 0 & \boldsymbol{e}^T \\ \boldsymbol{e} & \boldsymbol{O} \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} = \bar{N}$$
$$\begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{e}\boldsymbol{e}^T \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} = \bar{N}^2$$
$$\begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{A} \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} \le 2\theta N^2$$
$$Y_{ii} = 1$$
$$\begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} \succeq \boldsymbol{O}.$$

$$OPT_{SOCP} = \max_{\boldsymbol{x}\in\mathbb{R}^Z} \; : \; \boldsymbol{g}^T\boldsymbol{x}$$
$$\text{s.t.} \; : \; \boldsymbol{e}^T\boldsymbol{x} = 1, \quad ||\boldsymbol{B}\boldsymbol{x}|| \le \sqrt{2\theta}, \quad 0 \le x_i \le \frac{1}{N}$$

$$OPT_{LP} = \max \; : \; \begin{pmatrix} 0 & \hat{\boldsymbol{g}}^T \\ \hat{\boldsymbol{g}} & \boldsymbol{O} \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} + \bar{g}$$
$$\text{s.t.} \; : \; \begin{pmatrix} 0 & \boldsymbol{e}^T \\ \boldsymbol{e} & \boldsymbol{O} \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} = \bar{N}$$
$$\begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{e}\boldsymbol{e}^T \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} = \bar{N}^2$$
$$\begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{A} \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} \le 2\theta N^2$$
$$Y_{ii} = 1$$
$$-1 \le y_i \le 1$$
$$-1 \le Y_{ij} \le 1 \; (i \ne j).$$

1: Easy to show $OPT_{ECP} \le OPT_{SDP} \le OPT_{LP}$

2: Some steps are necessary to prove $OPT_{ECP} \le OPT_{SDP} \le OPT_{SOCP}$

3: Some assumptions on input data are necessary for $OPT_{ECP} \le OPT_{SDP} \le OPT_{SOCP} \le OPT_{LP}$

due to the constraint $Y_{ii} = 1$

An SDP-based randomized algorithm for QCQP and its theoretical analysis

$$OPT_{QCQP} = \max_{\boldsymbol{y} \in \mathbb{R}^n} \; : \; \boldsymbol{y}^T \boldsymbol{A}^0 \boldsymbol{y} + (\boldsymbol{b}^0)^T \boldsymbol{y} + c^0$$
$$\text{s.t.} \; : \; \boldsymbol{y}^T \boldsymbol{A}^k \boldsymbol{y} + (\boldsymbol{b}^k)^T \boldsymbol{y} + c^k \le 0 (k = 1, \ldots, m)$$

$$OPT_{SDP} = \max \; : \; \boldsymbol{B}^0 \bullet \boldsymbol{Y}$$
$$\text{s.t.} \; : \; \boldsymbol{B}^k \bullet \boldsymbol{Y} \le 0 (k = 1, \ldots, m)$$
$$\boldsymbol{B}^{m+1} \bullet \boldsymbol{Y} = 1$$
$$\boldsymbol{Y} \succeq \boldsymbol{O}$$

$$\boldsymbol{B}^k = \begin{pmatrix} c^k & (\boldsymbol{b}^k)^T \\ \boldsymbol{b}^k & \boldsymbol{A}^k \end{pmatrix} (k = 0, 1, \ldots, m), \boldsymbol{B}^{m+1} = \begin{pmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{O} \end{pmatrix}$$

Cholesky factorization of SDP solution

$$\boldsymbol{Y}^* = \boldsymbol{V}^T \boldsymbol{V}, \qquad \boldsymbol{V} = [\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n]$$

Randomly-generated solution

$$\hat{\boldsymbol{y}} \in \mathbb{R}^n : \hat{y}_i = \sqrt{Y^*_{ii}} \text{sign}(\boldsymbol{v}^T \boldsymbol{v}_0) \text{sign}(\boldsymbol{v}^T \boldsymbol{v}_i) \; (i = 1, \ldots, n)$$

$\boldsymbol{v}$ is randomly chosen from $\{\boldsymbol{v} \in \mathbb{R}^n : ||\boldsymbol{v}|| = 1\}$.

Tseng (2003) analyzed the expected objective value

$$E[\hat{\boldsymbol{y}}^T \boldsymbol{A}^0 \hat{\boldsymbol{y}} + (\boldsymbol{b}^0)^T \hat{\boldsymbol{y}}] \ge \tfrac{2}{\pi} OPT_{SDP} + \left(1 - \tfrac{2}{\pi}\right) \rho_{SDP}$$

$$\rho_{SDP} = \min \; : \; \boldsymbol{B}^0 \bullet \boldsymbol{Y}$$
$$\text{s.t.} \; : \; \boldsymbol{B}^k \bullet \boldsymbol{Y} = \boldsymbol{B}^k \bullet \boldsymbol{Y}^* (k \in \mathcal{I})$$
$$\boldsymbol{B}^{m+1} \bullet \boldsymbol{Y} = 1$$
$$\boldsymbol{Y} \succeq \boldsymbol{O}$$

$$\mathcal{I} = \left\{ k \in \{1, 2, \ldots, m\} : \boldsymbol{B}^k \text{ is diagonal} \right\}$$

# A theoretical analysis of SDP relaxation for equally contribution problems

$$
\begin{aligned}
OPT_{SDP} = \max \quad : \quad & \begin{pmatrix} 0 & \hat{\boldsymbol{g}}^T \\ \hat{\boldsymbol{g}} & \boldsymbol{O} \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} + \bar{g} \\
\text{s.t.} \quad : \quad & \begin{pmatrix} 0 & \boldsymbol{e}^T \\ \boldsymbol{e} & \boldsymbol{O} \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} = \bar{N} \\
& \begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{e}\boldsymbol{e}^T \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} = \bar{N}^2 \\
& \begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{A} \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} \le 2\theta N^2 \\
& Y_{ii} = 1 \\
& \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \boldsymbol{Y} \end{pmatrix} \succeq \boldsymbol{O}.
\end{aligned}
$$

Generate $\hat{\boldsymbol{y}}$ by the randomized algorithm

Check the objective value $2\hat{\boldsymbol{g}}^T\hat{\boldsymbol{y}} + \bar{g}$

$$\frac{2}{\pi}OPT_{SDP} + \left(1 - \frac{2}{\pi}\right)\left(-2\hat{\boldsymbol{g}}^T\boldsymbol{e} + \bar{g}\right) \le E[2\hat{\boldsymbol{g}}^T\hat{\boldsymbol{y}} + \bar{g}] \le \alpha OPT_{SDP} + (1-\alpha)\left(2\hat{\boldsymbol{g}}^T\boldsymbol{e} + \bar{g}\right)$$

$$\alpha = \min\left\{\frac{2}{\pi}\frac{\theta}{1-\cos\theta} : 0 \le \theta \le \pi\right\} \sim 0.878$$

Z	2θ	lower bound	(Average of 1000 trials)	upper bound	OPT_{SDP}
200	0.0334	16.161	25.812	30.340	25.386
1050	0.0627	5.075	32.305	112.600	24.938
2045	0.0711	279.259	446.089	2007.212	438.659
5050	0.1081	5.775	284.965	806.205	42.786

Unfortunately, the bounds are not very sharp.
Most of the generated solutions are not feasible for the equally contribution problems.

$$\hat{\boldsymbol{x}}^T\boldsymbol{A}\hat{\boldsymbol{x}} > 2\theta$$

A maximization problem with a penalty term

- We need a method to obtain an approximate solution in a practical time
- We move the quadratic constraints into the objective function

$$
\begin{aligned}
OPT_{ECP} = \max_{\boldsymbol{x} \in \mathbb{R}^Z} \ &: \ \boldsymbol{g}^T \boldsymbol{x} \\
\text{s.t.} \ &: \ \boldsymbol{e}^T \boldsymbol{x} = 1 \\
&\quad \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} \leq 2\theta \\
&\quad x_i \in \left\{0, \frac{1}{N}\right\}
\end{aligned}
$$

$$
\begin{aligned}
OPT_{PNL} = \max_{\boldsymbol{x} \in \mathbb{R}^Z} \ &: \ \boldsymbol{g}^T \boldsymbol{x} - \lambda \max\{\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} - 2\theta, 0\} \\
\text{s.t.} \ &: \ \boldsymbol{e}^T \boldsymbol{x} = 1 \\
&\quad x_i \in \left\{0, \frac{1}{N}\right\}
\end{aligned}
$$

We can prove $OPT_{ECP} = OPT_{PEL}$ for sufficiently large λ.

For the weight λ,
we use the Lagrange multiplier corresponding to $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} = 2\theta$.

$$
\begin{aligned}
\max_{\boldsymbol{x} \in \mathbb{R}^Z} \ &: \ \boldsymbol{g}^T \boldsymbol{x} \\
\text{s.t.} \ &: \ \boldsymbol{e}^T \boldsymbol{x} = 1 \\
&\quad \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} = 2\theta
\end{aligned}
$$

Steep-ascent method

$$
\begin{aligned}
OPT_{PNL} = \max_{\boldsymbol{x} \in \mathbb{R}^Z} &: \quad f_\lambda(\boldsymbol{x}) := \boldsymbol{g}^T \boldsymbol{x} - \lambda \max\{\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} - 2\theta, 0\} \\
\text{s.t.} &: \quad \boldsymbol{e}^T \boldsymbol{x} = 1 \\
& \quad\ \ x_i \in \left\{0, \frac{1}{N}\right\}
\end{aligned}
$$

Idea: Search a better point by swapping $x_i = 0$ and $x_j = \frac{1}{N}$

1. Set $\boldsymbol{x}^+ \in \mathbb{R}^Z$ as an solution of a conic relaxation problem.
   Sort $\boldsymbol{x}^+$ and get the descending order s such that $x^+_{s(1)} \geq x^+_{s(2)} \geq x^+_{s(3)} \geq \ldots \geq x^+_{s(Z)}$.
   Set $\boldsymbol{x}^0 \in \{0, 1/N\}^Z$ such that $x^0_{s(i)} = 1/N$ for $i = 1, \ldots, N$ and $x^0_{s(i)} = 0$ for $i = N+1, \ldots, Z$.
   Set $V_0 = \{i \in \{1, \ldots, Z\} : x^0_i = 0\}$ and $V_{1/N} = \{i \in \{1, \ldots, Z\} : x^0_i = 1/N\}$.
   Set the iteration counter $h = 0$.

2. Find $(i^h, j^h) = \text{argmax}\left\{f_\lambda(\boldsymbol{x}^h + \frac{1}{N}\boldsymbol{e}_i - \frac{1}{N}\boldsymbol{e}_j) \quad \middle| \quad (i, j) \in V_0 \times V_{1/N}\right\}$.

3. If $f_\lambda(\boldsymbol{x}^h + \frac{1}{N}\boldsymbol{e}_{i^h} - \frac{1}{N}\boldsymbol{e}_{j^h}) \leq f_\lambda(\boldsymbol{x}^h)$, output $\boldsymbol{x}^h$ as the soltuion

4. Set $\boldsymbol{x}^{h+1} = \boldsymbol{x}^h + \frac{1}{N}\boldsymbol{e}_{i^h} - \frac{1}{N}\boldsymbol{e}_{j^h}$, $V_0 = V_0 \cup \{j_h\} \backslash \{i_h\}$, $V_{1/N} = V_{1/N} \cup \{i_h\} \backslash \{j_h\}$,
   $h \leftarrow h + 1$ and return to Step 2.

Steep-ascent and discrete convex functions

- Our steep-ascent method is a modification of
  the steep-descent method for M-convex functions that was implemented in ODICON.
  http://www.misojiro.t.u-tokyo.ac.jp/~tutimura/odicon/
- Unfortunately, our objective function is not an M-convex function.
  However, *the steep-ascent method finds at least a local optimizer*.

$$
\begin{aligned}
OPT_{PNL} = \max_{\boldsymbol{x} \in \mathbb{R}^z} \quad &: \quad f_\lambda(\boldsymbol{x}) := \boldsymbol{g}^T \boldsymbol{x} - \lambda \max\{\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} - 2\theta, 0\} \\
\text{s.t.} \quad &: \quad \boldsymbol{e}^T \boldsymbol{x} = 1 \\
& \quad\quad x_i \in \left\{0, \frac{1}{N}\right\}
\end{aligned}
$$

- By specializing the algorithm of ODICON in this specific problem, our implementation is 20 times faster than ODICON.

Core i7 3770K, 32GB
LP by CPLEX, SOCP by ECOS, SDP by SDPT3

Comparison of three conic relaxations

Algorithm	Z	2θ	$g^T x$	$x^T A x$	iter	time (s)
CR (LP)	200	0.0334	28.068	0.0574	0	0.07
SA (LP)			25.029	0.0334	21	0.10
CR (SOCP)			26.156	0.0334	0	0.02
SA (SOCP)			25.090	0.0334	13	0.06
CR (SDP)			25.386	0.0321	0	1.29
SA (SDP)			25.207	0.0334	4	1.30
CR (LP)	5050	0.1081	57.630	0.3672	0	10.17
SA (LP)			38.696	0.1080	23	11.17
CR (SOCP)			43.036	0.1081	0	0.21
SA (SOCP)			42.691	0.1080	3	0.37
CR (SDP)			42.786	0.0980	0	2221.22
SA (SDP)			42.431	0.1080	3	2221.40
CR (LP)	15222	0.0388	603.783	0.4568	0	129.55
SA (LP)			438.791	0.0388	42	139.03
CR (SOCP)			468.367	0.0388	0	0.99
SA (SOCP)			460.769	0.0388	9	2.56
CR (SDP)			288.739	0.0195	0	17433.38†
SA (SDP)			460.409	0.0388	43	17441.93†

- SDP attains the best approximation
- However, numerically instable (due to lack of interior-point)

- SOCP is much fast
- When combining with the steep-ascent, SOCP is competitive with SDP

N = 50
CR: convex relaxation (the solution may not be feasible for equally contribution problem)
SA: the steep-ascent method starting from the conic relaxation

Comparison with other existing methods

N = 50

Algorithm	Z	2θ	$g^T x$	$x^T A x$	$f_\lambda(x)$	#chosen	time (s)
GENCONT	200	0.0334	25.290	0.0342	20.087	50	0.06
OPSEL			25.191	0.0334	25.191	50	1779.13
CPLEX			25.190	0.0334	25.190	50	4270.77
SA (SOCP)			25.090	0.0334	25.090	50	0.06
GENCONT	1050	0.0627	24.983	0.0627	24.983	48	7.91
OPSEL			24.858	0.0627	24.858	50	> 10800
CPLEX			Cannot obtain a feasible solution in 3 hours				> 10800
SA (SOCP)			24.831	0.0627	24.831	50	0.09
GENCONT	5050	0.1081	42.780	0.1089	-306.701	50	1769.72
OPSEL			42.702	0.1081	42.702	50	> 10800
CPLEX			42.456	0.1066	42.456	50	2.02
SA (SOCP)			42.691	0.1080	42.691	50	0.37
GENCONT	10100	0.0701	Out of memory				
OPSEL			46.252	0.0700	46.252	50	> 10800
CPLEX			Cannot obtain a feasible solution in 3 hours				> 10800
SA (SOCP)			46.568	0.0701	46.568	50	0.87

The steep ascent method with SOCP outputs favorable solutions in very short time.

We stopped OPSEL and CPLEX when [gap < 1%] or [time > 3 hours].

Conclusion

- Optimal contribution problems in tree breeding
- Conic (LP, SOCP, SDP) relaxation problems
- Steep ascent method
- SOCP relaxation with the steep ascent method outputs a favorable solution in a practical time.

- Can we tighten the SDP relaxation with a shorter time?
- Other optimization problems?

- *Thank you very much for your attention!!*