An upper bound for the number of different solutions generated by the primal simplex method with any selection rule of entering variables

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Abstract

Kitahara and Mizuno (2011a) obtained an upper bound for the number of different solutions generated by the primal simplex method with Dantzig’s (the most negative) pivoting rule. In this paper, we obtain an upper bound with any pivoting rule which chooses an entering variable whose reduced cost is negative at each iteration. The bound is applied to special linear programming problems. We also get a similar bound for the dual simplex method.

Keywords: Linear programming; the number of basic solutions; pivoting rule; the simplex method.

1 Introduction

The simplex method for solving linear programming problems (LPs) was originally developed by Dantzig (1963). In spite of its practical efficiency, a good bound for the number of iterations of the simplex method has not been known for a long time. A main reason for this is a pessimistic result

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by Klee and Minty (1972). They showed that the simplex method needs an exponential number of iterations for a well-designed LP.

Kitahara and Mizuno (2011a) obtained an upper bound for the number of different basic feasible solutions generated by Dantzig’s simplex method (the primal simplex method with the most negative pivoting rule) for a standard form linear programming problem. The bound is

$$n\left[m\frac{\gamma P}{\delta P} \log(m\frac{\gamma P}{\delta P})\right],$$

where $m$ is the number of constraints, $n$ is the number of variables, $\delta P$ and $\gamma P$ are the minimum and the maximum values of all the positive elements of primal basic feasible solutions, and $\lceil r \rceil$ is the minimum integer bigger than a real number $r$. The number of different basic feasible solutions generated by the simplex method is not identical to the number of iterations in general, but it is identical if the primal problem is nondegenerate.

Kitahara, Matsui, and Mizuno (2011) improved the bound to

$$(n - m)\left[\min\{m, n - m\}\frac{\gamma P}{\delta P} \log(\min\{m, n - m\}\frac{\gamma P}{\delta P})\right].$$

These are restricted results since they are valid only for the most negative pivoting rule and the best improvement pivoting rule. In this paper, we handle any pivoting rule which chooses an entering variable whose reduced cost is negative at each iteration. We show that the number of different basic feasible solutions generated by the primal simplex method is at most

$$\lceil\min\{m, n - m\}\frac{\gamma P}{\delta P} \cdot \delta_D'\rceil,$$  \quad (1)

where $\delta_D$ and $\gamma_D'$ are the minimum and the maximum values of absolute values of all the negative elements of basic solutions of the dual problem. We also get a similar bound for the dual simplex method.

When the bound (1) is applied to a variant of Klee-Minty’s LP proposed by Kitahara and Mizuno (2011b), the upper bound (1) becomes

$$m(2^m - 1).$$

The ratio between this upper bound and the lower bound ($2^m - 1$) given in Kitahara and Mizuno (2011b) is much smaller than the bound itself.

In the case where the constraint matrix $A$ is totally unimodular and the constraint vector $\mathbf{b}$ and the objective vector $\mathbf{c}$ are integral, the bound becomes

$$\min\{m, n - m\}\|\mathbf{b}\|_1 \|\mathbf{c}\|_1.$$
2 The simplex method

In this section, we review the simplex method. We consider the primal LP

$$\begin{align*}
\min & \quad c^T x, \\
\text{subject to} & \quad Ax = b, \ x \geq 0,
\end{align*}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ are given data and $x \in \mathbb{R}^n$ is a variable vector. The dual problem of (2) is

$$\begin{align*}
\max & \quad b^T y, \\
\text{subject to} & \quad A^T y + s = c, \ s \geq 0,
\end{align*}$$

where $y \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$ are variable vectors. In this paper, we make the following assumptions.

1. $\text{rank}(A) = m$.
2. The primal problem (2) has an optimal solution.
3. An initial basic feasible solution $x^0$ of the primal problem is known.

Let $x^*$ be an optimal basic feasible solution of (2) and let $z^*$ be the optimal value. From the duality theorem, the dual problem (3) also has an optimal solution and the optimal value is $z^*$. Let $(y^*, s^*)$ be an optimal basic feasible solution of the dual problem (3).

We split $A$, $c$, $x$ and $s$ according to an index set $B \subset \{1, 2, \ldots, n\}$ and its complementary set $N = \{1, 2, \ldots, n\} - B$ like

$$A = (A_B, A_N), \quad c = \begin{bmatrix} c_B \\ c_N \end{bmatrix}, \quad x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}, \quad s = \begin{bmatrix} s_B \\ s_N \end{bmatrix}.$$ 

We call $B$ a basis when $A_B$ is an $m \times m$ nonsingular matrix. Let $B$ be the set of bases. For any basis $B \in \mathcal{B}$ and nonbasis $N = \{1, 2, \ldots, n\} - B$, the primal problem can be written as

$$\begin{align*}
\min & \quad c_B^T x_B + c_N^T x_N, \\
\text{subject to} & \quad A_B x_B + A_N x_N = b, \ x \geq 0.
\end{align*}$$

Since $A_B$ is nonsingular, we further rewrite the problem as

$$\begin{align*}
\min & \quad c_B^T A_B^{-1} b + (c_N - A_N^T (A_B^{-1})^T c_B) x_N, \\
\text{subject to} & \quad x_B = A_B^{-1} b - A_B^{-1} A_N x_N, \\
x_B \geq 0, & \quad x_N \geq 0.
\end{align*}$$

(4)
This form is called a dictionary for the primal problem (2) with \( B \). From the dictionary, we can get the basic solution

\[
x^B = \begin{bmatrix} x^B_B \\ x^B_N \end{bmatrix}, \quad x^B_B = A^{-1}_B b, \quad x^B_N = 0.
\]

(5)

If \( x^B_B \geq 0 \), this is a basic feasible solution. We define the set of primal feasible bases by \( \mathcal{B}_P = \{ B \in \mathcal{B} | x^B_B \geq 0 \} \). Let \( \delta_P \) and \( \gamma_P \) be the minimum and the maximum values of all the positive elements of primal basic feasible solutions, respectively. Then we have

\[
\delta_P \leq x^B_j \leq \gamma_P \text{ if } B \in \mathcal{B}_P \text{ and } x^B_j \neq 0.
\]

(6)

Similarly, the dual problem (3) can be written as

\[
\begin{align*}
\max \quad & b^T y, \\
\text{subject to} \quad & A^T_B y + s_B = c_B, \\
& A^T_N y + s_N = c_N, \\
& s_B \geq 0, \quad s_N \geq 0,
\end{align*}
\]

and we can proceed further:

\[
\begin{align*}
\max \quad & b^T (A^T_B)^{-1} c_B - b^T (A^T_B)^{-1} s_B, \\
\text{subject to} \quad & y = (A^T_B)^{-1} c_B - (A^T_B)^{-1} s_B, \\
& s_N = (c_N - A^T_N (A^T_B)^{-1} c_B) + A^T_N (A^T_B)^{-1} s_B, \\
& s_B \geq 0, \quad s_N \geq 0,
\end{align*}
\]

(7)

This is called a dictionary for the dual problem (3) with \( B \). From the dictionary, we can obtain the basic solution

\[
y^B = (A^T_B)^{-1} c_B, \quad s^B = \begin{bmatrix} s^B_B \\ s^B_N \end{bmatrix}, \quad s^B_B = 0, \quad s^B_N = c_N - A^T_N (A^T_B)^{-1} c_B.
\]

(8)

If \( s^B_N \geq 0 \), this is a dual basic feasible solution. We define the set of dual feasible bases by \( \mathcal{B}_D = \{ B \in \mathcal{B} | s^B_N \geq 0 \} \).

Let \( \delta_D \) and \( \gamma_D \) be the minimum and the maximum values of all the positive elements of \( s^B \) for any dual basic feasible solution \( (y^B, s^B) \). Then we have

\[
\delta_D \leq s^B_j \leq \gamma_D \text{ if } B \in \mathcal{B}_D \text{ and } s^B_j \neq 0.
\]

(9)

In addition, we define

\[
\delta'_D = \min \{-s^B_j | B \in \mathcal{B}_P \text{ and } s^B_j < 0\}
\]
and
\[ \gamma_D' = \max\{-s_j^B | B \in B_P \text{ and } s_j^B < 0\}. \]

From these definitions we have
\[ -\gamma_D' \leq s_j^B \leq -\delta_D' \text{ if } B \in B_P \text{ and } s_j^B < 0. \] (10)

We also define
\[ \delta_P' = \min\{-x_j^B | B \in B_D \text{ and } x_j^B < 0\} \]
and
\[ \gamma_P' = \max\{-x_j^B | B \in B_D \text{ and } x_j^B < 0\}. \]

Then we have
\[ -\gamma_P' \leq x_j^B \leq -\delta_P' \text{ if } B \in B_D \text{ and } x_j^B < 0. \] (11)

We remark that the values of \( P, P', \delta_P', \) and \( \gamma_P' \) depend only on \( A \) and \( b \) but not on \( c \), while the values \( \delta_D, \gamma_D, \delta_D', \) and \( \gamma_D' \) depend only on \( A \) and \( c \) but not on \( b \).

The primal dictionary (4) can be expressed by using the primal basic solution (5) and the dual basic solution (8):

\[
\begin{align*}
\text{min} & \quad z_B + (s_B^N)^T x_N, \\
\text{subject to} & \quad x_B = x_B^B - A_B^{-1} A_N x_N, \\
& \quad x_B \geq 0, \quad x_N \geq 0,
\end{align*}
\]

where \( z_B = c_B^T A_B^{-1} b \). Similarly, the dual dictionary (7) can be written as

\[
\begin{align*}
\text{max} & \quad z_B - (x_B^B)^T s_B, \\
\text{subject to} & \quad y = y^B - (A_B^{-1})^T s_B, \\
& \quad s_N = s_N^B + A_N^T (A_B^T)^{-1} s_B, \\
& \quad s_B \geq 0, \quad s_N \geq 0.
\end{align*}
\]

Since the objective function is independent of \( y \), we can ignore \( y \) and obtain the following LP

\[
\begin{align*}
\text{max} & \quad z_B - (x_B^B)^T s_B, \\
\text{subject to} & \quad s_N = s_N^B + A_N^T (A_B^T)^{-1} s_B, \\
& \quad s_B \geq 0, \quad s_N \geq 0. \quad (12)
\end{align*}
\]

Suppose that we generate a sequence of primal basic feasible solutions \( \{x^k | k = 1, 2, \ldots\} \) and a sequence of bases \( \{B^k \in B_P | k = 1, 2, \ldots\} \) by the primal simplex method from an initial basic feasible solution \( x^0 \). Let
$B^k \in B_P$ be the $k$-th basis and $N^k = \{1, 2, \ldots, n\} - B^k$. Then the dictionary (4) can be expressed as

$$
\begin{align*}
\min & \quad z^{B^k} + (s_{N^k}^{B^k})^T x_{N^k}, \\
\text{subject to} & \quad x_{B^k} = x_{B^k}^{B^k} - A_{B^k}^{-1} A_{N^k} x_{N^k}, \\
& \quad x_{B^k} \geq 0, \ x_{N^k} \geq 0.
\end{align*}
$$

If $s_{N^k}^{B^k} \geq 0$ holds, the current basic feasible solution $x^k$ is optimal. Otherwise we conduct a pivot. In this paper, we consider the primal simplex method which choose an entering variable whose reduced cost is negative. Thus we choose a nonbasic variable (entering variable) $x_j$ such that $s_j^{B^k} < 0$ and $j \in N^k$. Then the value of the entering variable is increased until a basic variable becomes zero.

There are several rules to decide an entering variable, for example, the minimum coefficient rule (Dantzig’s rule), the best improvement rule, and the minimum index rule. Under Dantzig’s rule, we choose an index according to

$$
j^k \in \arg \min_{j \in N^k} s_j^{B^k}
$$

and set $x_{j^k}$ as an entering variable.

We briefly explain the dual simplex method. Let $(y^k, s^k)$ and $B^k \in B_D$ be the $k$-th iterate and the $k$-th basis. Then from (12), the $k$-th dictionary is expressed as

$$
\begin{align*}
\max & \quad z^{B^k} - (x_{B^k}^{B^k})^T s_{B^k}, \\
\text{subject to} & \quad s_{N^k} = s_{N^k}^{B^k} + A_{N^k}^T (A_{B^k}^T)^{-1} s_{B^k}, \\
& \quad s_{B^k} \geq 0, \ s_{N^k} \geq 0.
\end{align*}
$$

Unless $x_{B^k}^{B^k} \geq 0$, we conduct a pivot. We choose a basic variable $s_j$, $j \in B^k$ which satisfies $x_j^{B^k} < 0$, then we increase the value of $s_j$.

3 A new bound for the primal simplex method

First we show that for any nonoptimal basic feasible solution $x^t$ of (2), we can obtain a lower bound of the optimal value.

**Lemma 1** We have

$$
z^* \geq c^T x^t - \min\{n - m, m\} \gamma_P \gamma_D',
$$

where $z^*$ is the optimal value of (2).
Proof: Let \( x^* \) be an optimal basic feasible solution. We have
\[
  z^* = c^T x^* \\
  = z_B + (s_{N_1}^B)^T x^*_N \\
  \geq c^T x' - \gamma_D e^T x^*_N \\
  \geq c^T x' - \min\{n - m, m\} \gamma_D 
\]
The first inequality follows from (10). The second inequality follows from the facts that \(|N'| = n - m\), the number of positive elements of \( x^* \) is at most \( m \), and (6). The desired inequality readily follows from the above inequality. We show that the objective value decreases at least by a constant value when the primal simplex method updates the iterate.

Theorem 1 Let \( x^t \) and \( x^{t+1} \) be the \( t \)-th and \((t+1)\)-st iterates generated by the primal simplex method. If \( x^t \) is not optimal and \( x^{t+1} \neq x^t \), we have
\[
  c^T x^t - c^T x^{t+1} \geq \delta_D 
\]

Proof: Let \( x_{j^t} \) be the entering variable at the \( t \)-th iteration. If \( x_{j^t}^{t+1} = 0 \) we have \( x^{t+1} = x^t \), which contradicts the assumption. Thus \( x_{j^t}^{t+1} \neq 0 \) which implies \( x_{j^t}^{t+1} \geq \delta_D \) from (6), and \( s_{j^t}^i \leq -\delta_D \) from (10). Then we have
\[
  c^T x^t - c^T x^{t+1} = -s_{j^t}^i x_{j^t}^{t+1} \geq \delta_D \delta_P. 
\]
Thus the proof is completed. From Lemma 3.1 and Theorem 3.1, we obtain an upper bound for the number of different basic feasible solutions generated by the primal simplex method.

Corollary 1 Suppose that we generate a sequence of basic feasible solutions by the primal simplex method from an initial iterate \( x^0 \). Then the number of different basic feasible solutions is at most
\[
  \left\lceil \min\{m, n - m\} \gamma_D \gamma_P \left\lceil \frac{1}{\delta_D \delta_D} \right. \right\rceil 
\]
where \( \lceil r \rceil \) is the minimum integer bigger than a real value \( r \).
4 Applications to special LPs

4.1 Klee-Minty’s LP

Kitahara and Mizuno (2011b) constructed a variant of Klee-Minty’s LP. For a given integer $m \geq 2$, the problem is represented as

$$\begin{align*}
\text{max} & \quad \sum_{i=1}^{m} x_i, \\
\text{subject to} & \quad x_1 \leq 1, \\
& \quad 2 \sum_{i=1}^{k-1} x_i + x_k \leq 2^k - 1 \text{ for } k = 2, 3, \ldots, m, \\
& \quad x = (x_1, x_2, \ldots, x_m)^T \geq 0.
\end{align*}$$

By introducing a slack vector $y = (y_1, y_2, \ldots, y_m)^T$, the problem is equivalent to

$$\begin{align*}
\text{max} & \quad \sum_{i=1}^{m} x_i, \\
\text{subject to} & \quad x_1 + y_1 = 1, \\
& \quad 2 \sum_{i=1}^{k-1} x_i + x_k + y_k = 2^k - 1 \text{ for } k = 2, 3, \ldots, m, \\
& \quad x \geq 0, \quad y \geq 0.
\end{align*}$$

(13)

Kitahara and Mizuno (2011b) show that the variant (13) has the following properties.

1. The number of basic feasible solutions is $2^m$.
2. The number of different basic feasible solutions generated by Dantzig’s primal simplex method is $(2^m - 1)$.
3. We have

$$\begin{align*}
\delta_P &= 1, \quad \gamma_P = 2^m - 1, \quad \delta'_D = 1, \quad \gamma'_D = 1
\end{align*}$$

from Theorem 2.1 in Kitahara and Mizuno (2011b).

By using the bound in Corollary 3.1 and $n = 2m$, the number of different basic feasible solutions is at most

$$|\min\{m, n - m\} \frac{\gamma_P \gamma'_D}{\delta_P \delta'_D}| = m(2^m - 1) + 1.$$  

(14)

It is not hard to show that +1 in (14) is not actually required. The ratio between this upper bound and the lower bound $(2^m - 1)$ given in Kitahara and Mizuno (2011b) is about $m$, which is much smaller than the bound itself.
4.2 LP with totally unimodular matrix

A matrix \( A \in \mathbb{R}^{m \times n} \) is said to be totally unimodular if the determinant of every square submatrix of \( A \) is \( \pm 1 \) or 0. In this subsection, we assume that the constraint matrix \( A \) is totally unimodular and the constraint vector \( b \) and the objective vector \( c \) are integral in (2). Then all the elements of any primal or dual basic solution are integers, which means that \( \delta_P = \delta'_D = 1 \). Let \( B \subset \{1, 2, \ldots, n\} \) be a basis. Then the basic solution associated with \( B \) is \( x^B_B = (A_B)^{-1}b \), \( x^B_N = 0 \). Since \( A \) is totally unimodular, all the elements of \( A_B^{-1} \) is \( \pm 1 \) or 0. Thus for any \( j \in B \), we have

\[
|x^B_j| \leq \|b\|_1,
\]

which implies \( \gamma_P, \gamma'_P \leq \|b\|_1 \). Similarly, we can show that \( \gamma_D, \gamma'_D \leq \|c\|_1 \). Hence we get the next theorem.

**Theorem 2** Suppose that the constraint matrix \( A \) is totally unimodular and the constraint vector \( b \) and the objective vector \( c \) are integral. Then the number of different solutions generated by the primal simplex method for solving (2) is at most

\[
\min\{m, n - m\}\|b\|_1\|c\|_1 + 1 .
\]

(15)

It is not hard to show that +1 in (15) is not actually required.

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