Buyback Problem with Discrete Concave Valuation Functions☆

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Abstract

We discuss an online discrete optimization problem called the buyback problem. In the literature of the buyback problem, the valuation function representing the total value of selected elements is given by a linear function. In this paper, we consider a generalization of the buyback problem using nonlinear valuation functions. We propose an online algorithm for the problem with discrete concave valuation functions, and show that it achieves the tight competitive ratio, i.e., the competitive ratio of the proposed algorithm is equal to the known lower bound for the problem.

Keywords: buyback problem, discrete concave function, gross substitutes valuation, online discrete optimization problem, matroid

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1. Introduction

We discuss an online discrete optimization problem called the buyback problem. In the literature of the buyback problem, the valuation function representing the total value of selected elements is given by a linear (or additive) function. We refer to this variant of the buyback problem as the *linear buyback problem*. In this paper, we consider the *nonlinear buyback problem*, a generalization of the buyback problem with a nonlinear valuation function.

1.1. Model of Nonlinear Buyback Problem

To explain the setting of the nonlinear buyback problem, we consider a situation where a company wants to hire some workers from a finite set N of applicants. Each applicant arrives one by one sequentially, and an interviewer of the company, which corresponds to an online algorithm, must decide immediately whether or not to hire the applicant. The company can hire at most m applicants; in addition, there may be some other constraints for a set of hired applicants due to their job skills and/or their human relationship. We denote by $\mathcal{F} \subseteq 2^N$ the set of feasible combinations of applicants. The interviewer wants to maximize the profit $v(X)$ obtained from a set $X \in \mathcal{F}$ of hired applicants. The function v is a nonlinear function in X in general since the job skill of applicants may overlap. It is natural to assume that function v is monotone nondecreasing and satisfies $v(\emptyset) = 0$. It is often the case that a good applicant comes for an interview but addition of the applicant violates the feasibility. In such a case, the interviewer can add the applicant by canceling the contract with some previously hired applicant at the cost of compensatory payment. In this paper, we assume that cancellation cost is given by a constant $c > 0$. We assume that if an applicant is rejected at the interview or once accepted but canceled later, then the applicant cannot be recovered later. Thus, the payoff obtained by the company is given as the value $v(B)$ of hired applicants B (at the end of the interviews) minus the total cancellation cost. The goal of the interviewer is to make an online decision to maximize the payoff.

Formally, the (nonlinear) buyback problem is formulated as an online version of the following discrete optimization problem:

$$\text{Maximize } v(B) - c|C| \qquad \text{subject to } B \in \mathcal{F},\ C \subseteq N,\ B \cap C = \emptyset,$$

where B is a set of hired applicants at the end of the interviews and C is a set of applicants who are once accepted but later canceled. It is assumed that the set family $\mathcal{F}$ and the function v are accessible via appropriate oracles;

that is, for a given set $X \subseteq N$, whether $X \in \mathcal{F}$ or not can be checked in constant time, and if $X \in \mathcal{F}$ then the function value $v(X)$ can be obtained in constant time.

The special case of the buyback problem with a linear valuation function given as $v(X) = \sum_{i \in X} w(i)$ and a matroid constraint is discussed by Kawase, Han, and Makino [18], who proposed an online algorithm and analyzed its competitive ratio. A *competitive ratio* of an online algorithm for the buyback problem is an upper bound of the ratio

$$\frac{v(B^*)}{v(B) - ct}$$

for all possible instances of the problem, where B is a set of hired applicants computed by the online algorithm, t is the number of cancellation made by the online algorithm, and B^* is an (offline) optimal solution of the instance. It is assumed that a value $\ell > 0$ with $\ell \leq \min_{i \in N} w(i)$ is known in advance, and let $r^*(\ell, c)$ be a positive real number given by

$$r^*(\ell, c) = 1 + \frac{c + \sqrt{c^2 + 4\ell c}}{2\ell}. \tag{1.1}$$

Note that the value $r^*(\ell, c)$ is dependent only on the ratio ℓ/c; see Figure 1 for a graph showing the relation between $r^*(\ell, c)$ and ℓ/c. For example, if $\ell/c = 2$ then $r^*(\ell, c) = 2$, and if $\ell/c = 6$ then $r^*(\ell, c) = 1.5$.

Theorem 1.1 ([18]). *Suppose that $v : 2^N \to \mathbb{R}$ is a linear valuation function and $\mathcal{F} \subseteq 2^N$ is the family of independent sets of a matroid. Then, the buyback problem admits an online algorithm with the competitive ratio $r^*(\ell, c)$. Moreover, there exists no online deterministic algorithm with a competitive ratio smaller than $r^*(\ell, c)$, even in the special case with $\mathcal{F} = \{X \subseteq N \mid |X| \leq 1\}$.*

The main aim of this paper is to generalize this result to the buyback problem with discrete concave valuation functions.

1.2. Our Result

In this paper, we present the first online algorithm for the nonlinear buyback problem and analyze its competitive ratio theoretically. In particular, we show that our algorithm achieves the tight upper bound for the competitive ratio that matches the lower bound in Theorem 1.1.

Our main results given in Theorems 1.2 and 1.3 are proved in Section 4 by generalizing the approach used in [18] for the linear buyback problem.

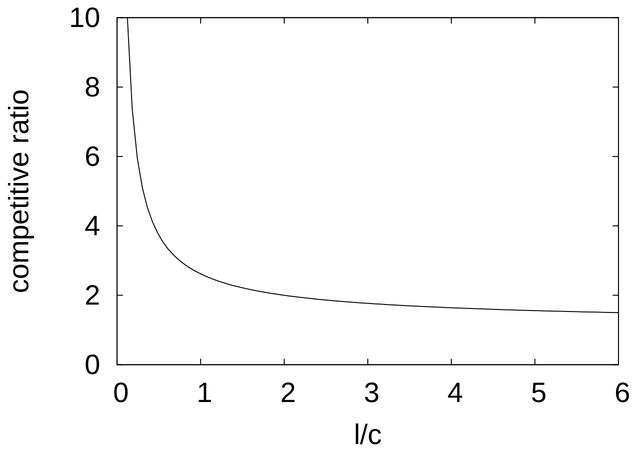

Figure 1: Relation between the values $r^*(\ell, c)$ and ℓ/c.

The analysis of the competitive ratio in our setting, however, is much more difficult due to the nonlinearity of valuation function. We overcome this difficulty by utilizing discrete concavity of the function called M$^\natural$-concavity. M$^\natural$-concavity of the valuation function plays a crucial role in the analysis for competitive ratio of our online algorithm. It should be noted that while M$^\natural$-concave functions satisfy a variant of submodular inequality, submodularity alone is not enough to obtain the current result; see Concluding Remarks.

1.2.1. Buyback Problem with Gross Substitutes Valuations and Matching Weight Valuations

We first consider a nonlinear valuation function called a gross substitutes valuation. A valuation function $v : 2^N \to \mathbb{R}$ defined on 2^N is called a *gross substitutes valuation* (*GS valuation*, for short) if it satisfies the following condition:

$$\forall p, q \in \mathbb{R}^N \text{ with } p \leq q, \ \forall X \in \arg\max_{U \subseteq N} \{v(U) - \sum_{i \in U} p(i)\},$$
$$\exists Y \in \arg\max_{U \subseteq N} \{v(U) - \sum_{i \in U} q(i)\} \text{ such that } \{i \in X \mid p(i) = q(i)\} \subseteq Y.$$

Intuitively, this condition is understood as follows, where N is regarded as a set of discrete items, and p and q are price vectors: if a buyer wants a set X of items at price p but some of the item prices are increased, then the buyer still wants items in X with unchanged prices (and possibly other items).

A natural but nontrivial example of GS valuations arises from the maximum-weight matching problem on a complete bipartite graph, called an *assignment valuation* [30] (or *OXS valuation* [21]). Going back to the situation at the company in Section 1.1, we suppose that the company has a set J of m jobs, to which hired workers are assigned. Each worker is assigned to at most one job in J, each job can be assigned to at most one worker, and if worker $i \in N$ is assigned to a job $j \in J$, then profit $q(i, j)$ given by a positive real number is obtained. Given a set $X \subseteq N$ of workers, the maximum total profit $v(X)$ obtained by assigning workers in X to jobs in J can be formulated as the maximum-weight matching problem on a complete bipartite graph G with the vertex sets N and J:

$$v(X) = \max\left\{ \sum_{(i,j)\in M} q(i,j) \;\middle|\; M : \text{matching in } G \text{ s.t. } \partial_N M = X \right\}, \quad (1.2)$$

where $\partial_N M$ denotes the set of vertices in N covered by edges in M. It is known that the function $v : 2^N \to \mathbb{R}$ is a GS valuation function [21, 30].

The concept of GS valuation is introduced by Kelso and Crawford [19], where the existence of a Walrasian equilibrium is shown in a fairly general two-sided matching model. Since then, this concept is widely used in various economic models and plays a central role in mathematical economics and in auction theory (see, e.g., [5, 6, 11, 13, 14, 21]). The class of GS valuations is a proper subclass of submodular functions, and includes natural classes of valuations such as weighted rank functions of matroids [7, 9] and laminar concave functions [25] (or *S-valuations* [5]), in addition to assignment valuations explained above. While GS valuation is a sufficient condition for the existence of a Walrasian equilibrium [19], it is also a necessary condition in some sense [14]. GS valuation is also related to desirable properties in the auction design [6, 11, 21]. See [28, 33] for more details on GS valuations as well as other related concepts.

We propose an online algorithm for the nonlinear buyback problem with a GS valuation function and a cardinality constraint. We assume that a positive real number ℓ satisfying

$$\ell \leq \min\{v(X)/|X| \mid \emptyset \neq X \in \mathcal{F}\} \quad (1.3)$$

is known in advance. Note that this condition is a natural generalization of the condition used in [18]; indeed, for a linear valuation function $v(X) = \sum_{i\in X} w(i)$, condition (1.3) is simply rewritten as $\ell \leq \min_{i\in N} w(i)$. In addition, if v is an assignment valuation function in (1.2), then every ℓ with $\ell \leq \min\{q(i, j) \mid i \in N,\ j \in J\}$ satisfies (1.3).

Theorem 1.2. *For a gross substitutes valuation function $v : 2^N \to \mathbb{R}$ and a cardinality constraint $\mathcal{F} = \{X \subseteq N \mid |X| \leq m\}$, the nonlinear buyback problem admits an online algorithm with the competitive ratio $r^*(\ell, c)$ in* (1.1).

By Theorem 1.1, there exists no online deterministic algorithm with a competitive ratio smaller than $r^*(\ell, c)$. Hence, Theorem 1.2 shows that our online algorithm achieves the tight upper bound matching the lower bound.

It should be noted that our online algorithm does not require the information about the number of elements in N and the integer m. We also note that even if the value ℓ with (1.3) is not known in advance, we can obtain the same competitive ratio by slight modification of the proposed algorithm; see Concluding Remarks.

1.2.2. Buyback Problem with Discrete Concave Valuations

Moreover, we consider a more general setting where $\mathcal{F}$ is a matroid and valuation function $v : \mathcal{F} \to \mathbb{R}$ is a discrete concave function called *$M^\natural$-concave function.* A function $v : \mathcal{F} \to \mathbb{R}$ is said to be $\mathrm{M}^\natural$-concave [27] (read "M-natural-concave") if it satisfies a certain exchange axiom similar to that for matroid (see Section 2 for a precise definition of $\mathrm{M}^\natural$-concave function).

The concept of $\mathrm{M}^\natural$-concave function is introduced by Murota and Shioura [27] (independently of GS valuations) as a class of discrete concave functions. $\mathrm{M}^\natural$-concavity is originally introduced for functions defined on integer lattice points (see, e.g., [25]), and the present definition of $\mathrm{M}^\natural$-concavity for set functions can be obtained by specializing the original definition through the one-to-one correspondence between set functions and functions defined on $\{0, 1\}$-vectors. The concept of $\mathrm{M}^\natural$-concave function is an extension of the concept of M-concave function introduced by Murota [22, 24]. The concepts of $\mathrm{M}^\natural$-concavity/M-concavity play primary roles in the theory of discrete convex analysis [25], which provides a framework for tractable nonlinear discrete optimization problems.

$\mathrm{M}^\natural$-concave functions have various desirable properties as discrete concavity. Global optimality is characterized by local optimality, which implies the validity of a greedy algorithm for $\mathrm{M}^\natural$-concave function maximization. Maximization of an $\mathrm{M}^\natural$-concave function can be done efficiently in polynomial time (see, e.g., [25, 27]).

The class of $\mathrm{M}^\natural$-concave functions includes linear functions on matroids. Hence, the $\mathrm{M}^\natural$-concave buyback problem (i.e., the buyback problem with an $\mathrm{M}^\natural$-concave valuation function) is a proper generalization of the linear buyback problem with a matroid constraint discussed by Kawase et al. [18]. Furthermore, the $\mathrm{M}^\natural$-concave buyback problem also includes the problem

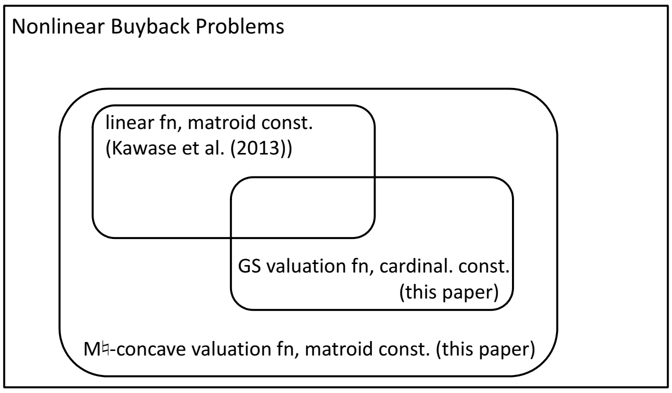

Figure 2: Relationship among the three classes of the buyback problems.

with a GS valuation function and a cardinality constraint as a special case. See Figure 2 for the relationship among the classes of buyback problems.

In this paper, we show the following result for the M$^\natural$-concave buyback problem.

Theorem 1.3. *If $\mathcal{F} \subseteq 2^N$ is the family of independent sets of a matroid and $v : \mathcal{F} \to \mathbb{R}$ is an M$^\natural$-concave function, then the nonlinear buyback problem admits an online algorithm with the competitive ratio $r^*(\ell, c)$ in* (1.1).

This theorem implies Theorem 1.2 as a corollary. In addition, this theorem also implies the former statement of Theorem 1.1, hence generalizing the result of Kawase et al. [18]. The latter statement in Theorem 1.1 shows that the competitive ratio in Theorem 1.3 is the best possible for the M$^\natural$-concave buyback problem, i.e., our online algorithm achieves the tight upper bound matching the lower bound.

1.3. Related Work

We review previous results on the linear buyback problem and some related problems. In the literature of the linear buyback problem, two types of cancellation cost are considered so far: *proportional* cost and *unit* cost; the latter one is used in this paper. In the case of proportional cost, we are given a constant $f > 0$ such that the cancellation cost of an element u with the value $w(u)$ is equal to $fw(u)$. In the case of unit cost, we are given a constant $c > 0$ such that the cancellation cost of each element u is equal to c. Note that in the nonlinear buyback problem, unit cancellation cost is more suitable since proportional cancellation cost is dependent on the linearity of a valuation function.

The linear buyback problem is originally modeled by using proportional cost. In this setting, Babaioff et al. [3] and Constantin et al. [10] independently proposed deterministic online algorithms for the problem with a single matroid constraint, where the competitive ratio is $1+2f+2\sqrt{f(1+f)}$. Babaioff et al. [4] also showed that this competitive ratio is the best possible bound for deterministic algorithms, and presented a randomized algorithm with a better competitive ratio in the case of small f. Later, Ashwinkumar and Kleinberg [2] proposed a randomized algorithm with an improved competitive ratio, which is shown to be the best possible. Ashwinkumar [1] considered a more general constraints such as the intersection of multiple matroids, and proposed online algorithms with theoretical bounds for the competitive ratio. Some variants of knapsack constraint were also considered in [3, 4, 15].

The linear buyback problem with unit cost was first introduced by Han et al. [15]. Some variants of knapsack constraints are considered in [15, 18], while single matroid constraint is considered by Kawase et al. [18] (see Theorem 1.1).

Variants of the buyback problem with zero cancellation cost are also extensively discussed in the literature. One such example is the problem under a knapsack constraint, which is referred to as the *online removable knapsack problem* (see, e.g., [16, 17]). Recently, the nonlinear buyback problem with zero cancellation cost and submodular valuation function (called the *online submodular maximization with preemption*) is considered by Buchbinder et al. [8]. Note that the linear buyback problem with a single matroid constraint is trivial if the cancellation cost is zero; indeed, existing online algorithms for this problem reduce to variants of greedy algorithms that find an (offline) optimal solutions.

The buyback problem with an assignment valuation function can be seen as a variant of online bipartite matching problems, where vertices on the one side of a bipartite graph (corresponding to applicants) arrive online one by one (see, e.g., [20] and the references therein). Among many variants of such online matching problems, our problem setting is different in the following two points. First, we allow re-assignment of previously accepted vertices to the vertices on the other side whenever a newly arrived vertex is accepted. Second, we allow exchange of a previously accepted vertex with a newly arrived vertex by paying a cancellation cost. Without a cancellation cost, our online matching problem is trivial since we allow re-assignment; indeed, it is easy to construct an online algorithm that finds an (offline) optimal matching under this setting.

2. M$^\natural$-concave Functions and GS Valuations

In this section we review the concept of M$^\natural$-concavity and its connection with GS valuation. In the following, we denote by $\mathbb{R}_+$ and $\mathbb{Z}_+$ the sets of nonnegative real numbers and nonnegative integers, respectively.

Consider a function $v : 2^N \to \mathbb{R} \cup \{-\infty\}$ defined on all subsets X of N such that $v(X)$ is a finite value or equal to $-\infty$ for every $X \in 2^N$. The *effective domain* of function v is defined as

$$\mathrm{dom}\, v = \{X \in 2^N \mid v(X) > -\infty\}.$$

Hence, function v can be regarded as a function defined on $\mathrm{dom}\, v$. We say that function v is monotone nondecreasing if $v(X) \leq v(Y)$ holds for every $X, Y \in \mathrm{dom}\, v$ with $X \subseteq Y$. Note that for a monotone nondecreasing function v, it is possible that $v(X) > -\infty = v(Y)$ holds for some $X, Y \in \mathrm{dom}\, v$ with $X \subsetneq Y$.

A function $v : 2^N \to \mathbb{R} \cup \{-\infty\}$ is said to be *M$^\natural$-concave* if $\mathrm{dom}\, v$ is nonempty and v satisfies the following condition:

(M$^\natural$-EXC) $\forall X, Y \in \mathrm{dom}\, v$, $\forall i \in X \setminus Y$,

$$v(X) + v(Y) \leq \max\left[v(X - i) + v(Y + i), \max_{j \in Y \setminus X}\{v(X - i + j) + v(Y + i - j)\}\right],$$

where $X - i + j$ is a short-hand notation for $(X \setminus \{i\}) \cup \{j\}$. We see from (M$^\natural$-EXC) that the effective domain $\mathcal{F} = \mathrm{dom}\, v$ of an M$^\natural$-concave function satisfies the following condition:[2]

(B$^\natural$-EXC) $\forall X, Y \in \mathcal{F}$, $\forall i \in X \setminus Y$, at least one of (i) and (ii) holds:
(i) $X - i \in \mathcal{F}$, $Y + i \in \mathcal{F}$,
(ii) $\exists j \in Y \setminus X$: $X - i + j \in \mathcal{F}$, $Y + i - j \in \mathcal{F}$.

It is known that a set family $\mathcal{F} \subseteq 2^N$ is the family of independent sets of a matroid if and only if $\mathcal{F}$ satisfies (B$^\natural$-EXC) and contains an empty set, i.e., $\emptyset \in \mathcal{F}$ (see, e.g., [34]). In this paper, we deal with M$^\natural$-concave functions such that $\mathrm{dom}\, v$ is the family of independent sets of a matroid.

It is known that every M$^\natural$-concave function is a submodular function in the following sense (cf. [25]), where we admit the inequality of the form $-\infty \geq -\infty$ for convenience.

[2] It is shown [27] that the condition (B$^\natural$-EXC) for a set family $\mathcal{F} \subseteq 2^N$ characterizes the concept of g-matroid by Tardos [34].

Proposition 2.1 ([25, Th. 6.19]). *Let $v : 2^N \to \mathbb{R}\cup\{-\infty\}$ be an M$^\natural$-concave function such that* $\mathrm{dom}\, v$ *is the family of matroid independent sets. Then, it holds that*

$$v(X) + v(Y) \geq v(X \cup Y) + v(X \cap Y) \qquad (\forall X, Y \in 2^N).$$

An M$^\natural$-concave function also satisfies the following property.

Proposition 2.2 ([27, Th. 4.2]). *Let $v : 2^N \to \mathbb{R}\cup\{-\infty\}$ be an M$^\natural$-concave function. For every $X, Y \in \mathrm{dom}\, v$ with $|X| = |Y|$ and $i \in X \setminus Y$, there exists some $j \in Y \setminus X$ such that*

$$v(X) + v(Y) \leq v(X - i + j) + v(Y + i - j).$$

Note that the sum of an M$^\natural$-concave function and a linear function is again an M$^\natural$-concave function, while the sum of two M$^\natural$-concave functions is not M$^\natural$-concave in general.

The next property shows the connection between M$^\natural$-concavity and gross substitute valuation. In particular, the property below implies that the buyback problem with a gross substitute valuation function is a special case of M$^\natural$-concave buyback problem.

Theorem 2.3 (cf. [13]). *Let $v : 2^N \to \mathbb{R}$ be a function defined on 2^N.*
(i) *v is a GS valuation function if and only if it is M$^\natural$-concave.*
(ii) *Suppose that v is a GS valuation function and let m be a nonnegative integer. Then, the function $v_m : 2^N \to \mathbb{R} \cup \{-\infty\}$ such that*

$$\mathrm{dom}\, v_m = \{X \in 2^N \mid |X| \leq m\}, \qquad v_m(X) = v(X) \ (X \in \mathrm{dom}\, v_m)$$

is an M$^\natural$-concave function.

A simple example of M$^\natural$-concave function is a linear function. For a vector $w \in \mathbb{R}^N_+$ and a family $\mathcal{F} \subseteq 2^N$ of matroid independent sets, the function $v : 2^N \to \mathbb{R} \cup \{-\infty\}$ such that

$$\mathrm{dom}\, v = \mathcal{F}, \qquad v(X) = \sum_{i \in X} w(i) \quad (X \in \mathrm{dom}\, v) \tag{2.1}$$

is an M$^\natural$-concave function; in particular, v is a GS valuation function if $\mathrm{dom}\, v = 2^N$. In Appendix we give some nontrivial examples of M$^\natural$-concave functions and GS valuation functions; see [25, 26] for more examples.

3. Our Online Algorithm

Given an M$^\natural$-concave function $v : 2^N \to \mathbb{R} \cup \{-\infty\}$ such that $\mathcal{F} = \operatorname{dom} v$ is the family of matroid independent sets, we consider M$^\natural$-concave buyback problem formulated as follows:

$$\text{Maximize } v(B) - c|C| \qquad \text{subject to } B \in \mathcal{F},\ C \subseteq N,\ B \cap C = \emptyset,$$

where B is a set of hired applicants at the end of the interviews and C is a set of applicants who are once accepted but later canceled. We assume that function v is monotone nondecreasing (i.e., $v(X) \le v(Y)$ holds for $X, Y \in \mathcal{F}$ with $X \subseteq Y$) and satisfies $v(\emptyset) = 0$. In this section, we propose an online algorithm for M$^\natural$-concave buyback problem.

We denote $n = |N|$, and assume that $N = \{i_1, i_2, \ldots, i_n\}$ and the elements in N arrive in this order. Our online algorithm is based on the following greedy algorithm for M$^\natural$-concave function maximization problem, i.e., the special case of M$^\natural$-concave buyback problem with zero cancellation cost. In each iteration, the greedy algorithm maintains a set $B_k \in \mathcal{F}$. In the k-th iteration, the algorithm adds an element i_k (i.e., it sets $B_k = B_{k-1} + i_k$) if $B_{k-1} + i_k \in \mathcal{F}$; recall that the valuation function v is assumed to be nondecreasing. Otherwise, the greedy algorithm tries to "cancel" some element j_k in B_{k-1} by replacing it with i_k, where the element j_k is chosen so that the value $v(B_{k-1} - i_k + j_k)$ is maximized. If the value $v(B_{k-1} - j_k + i_k)$ is larger than $v(B_{k-1})$, then the greedy algorithm replace j_k with i_k; otherwise, the algorithm keeps the current set B_{k-1} unchanged (i.e., it sets $B_k = B_{k-1}$).

Algorithm M$^\natural$Greedy
Step 0: Set $B_0 = \emptyset$.
Step 1: For each element i_k, $k = 1, 2, \ldots, n$, in order of arrival,
do the following:
[Case 1: $B_{k-1} + i_k \in \mathcal{F}$] Set $B_k = B_{k-1} + i_k$.
[Case 2: $B_{k-1} + i_k \notin \mathcal{F}$] Let $j_k \in B_{k-1}$ be an element satisfying

$$v(B_{k-1} - j_k + i_k) = \max\{v(B_{k-1} - j + i_k) \mid j \in B_{k-1}\}. \qquad (3.1)$$

If $v(B_{k-1} - j_k + i_k) > v(B_{k-1})$, then set $B_k = B_{k-1} - j_k + i_k$ ("cancel j_k"); otherwise, set $B_k = B_{k-1}$ ("reject i_k").
Step 2: Output B_n. $\square$

Proposition 3.1 (cf. [12, 31])**.** *For an M$^\natural$-concave function $v : 2^N \to \mathbb{R} \cup \{-\infty\}$ such that* $\operatorname{dom} v$ *is the family of matroid independent sets, the algorithm* M$^\natural$Greedy *outputs a maximizer of v.*

A rigorous proof of Proposition 3.1 is given in Appendix.

Our online algorithm is obtained by modifying M$^\natural$Greedy as follows. Recall that if we cancel some element, then we need to pay the cost $c > 0$. Hence, it is natural to allow cancellation if we can increase the function value of v sufficiently. That is, we replace some element j_k in B_{k-1} with i_k if the value $v(B_{k-1} - j_k + i_k)$ is large enough compared to $v(B_{k-1})$. To control the number of cancellations, we use an increasing sequence of real numbers $\psi(t)$ $(t = 1, 2, \ldots)$ as parameters, which will be determined later by using c and ℓ. We assume that $\psi(1) = 0$ and $\psi(t + 1) - \psi(t)$ is nondecreasing with respect to t.

Use of numbers $\psi(t)$ plays a key role in the analysis of competitive ratio in Section 4. This technique is already used by Kawase et al. [18] in the case of linear valuation functions; we generalize the technique in a nontrivial way so that it can be applied to nonlinear (M$^\natural$-concave) valuation functions.

A detailed description of the algorithm is as follows.

Algorithm M$^\natural$BP

Step 0: Set $B_0 = \emptyset$.

Step 1: For each element i_k, $k = 1, 2, \ldots, n$, in order of arrival, do the following:

[Case 1: $B_{k-1} + i_k \in \mathcal{F}$]

Set $B_k = B_{k-1} + i_k$.

[Case 2: $B_{k-1} + i_k \notin \mathcal{F}$]

Let t_{k-1} be the integer with

$$\psi(t_{k-1}) \leq v(B_{k-1}) - \ell \cdot |B_{k-1}| < \psi(t_{k-1} + 1).$$

Let $j_k \in B_{k-1}$ be an element satisfying (3.1).

If $v(B_{k-1} - j_k + i_k) - \ell \cdot |B_{k-1}| \geq \psi(t_{k-1} + 1)$, then set $B_k = B_{k-1} - j_k + i_k$ ("cancel j_k"); otherwise, set $B_k = B_{k-1}$ ("reject i_k").

Step 2: Output B_n. $\square$

Note that if the values $\psi(t + 1) - \psi(t)$ are sufficiently small for all t, then the cancellation occurs whenever $v(B_{k-1} - j_k + i_k) > v(B_{k-1})$ holds, implying that the behavior of the algorithm M$^\natural$BP coincides with that of the algorithm M$^\natural$Greedy. We also note that $B_k \in \mathcal{F}$ holds if B_k is set to $B_{k-1} - j_k + i_k$ since

$$v(B_{k-1} - j_k + i_k) \geq \psi(t_{k-1} + 1) + \ell \cdot |B_{k-1}| > -\infty.$$

Hence, we have $B_k \in \mathcal{F}$ for $k = 0, 1, \ldots, n$. The behavior of the algorithm, the monotonicity of function v, and the inequality (1.3) for ℓ imply that

$$\begin{aligned}
&v(B_0) \leq v(B_1) \leq v(B_2) \leq \cdots \leq v(B_n), \\
&0 \leq v(B_1) - \ell \cdot |B_1| \leq v(B_2) - \ell \cdot |B_2| \leq \cdots \leq v(B_n) - \ell \cdot |B_n|.
\end{aligned}$$

4. Analysis of Proposed Algorithm

In this section, we analyze the competitive ratio of the online algorithm proposed in Section 3.

4.1. Bounding the Optimal Value

Let $B^* \in \mathcal{F}$ be an (offline) optimal solution of M$^\natural$-concave buyback problem. That is, $B^* \in \arg\max\{v(B) \mid B \in \mathcal{F}\}$. To analyze the competitive ratio of the algorithm above, we bound the value of $v(B^*)$ from above.

We denote $m = \max\{|X| \mid X \in \mathcal{F}\}$. We will derive the following upper bound of $v(B^*)$.

Lemma 4.1. $v(B^*) \leq v(B_n) + m(\psi(t_{n-1} + 1) - \psi(t_{n-1}))$.

To prove Lemma 4.1, we first show that the value $v(B^*)$ can be bounded from above using the output B_n of the algorithm.

For two sets $B, B' \in \mathcal{F}$ with $|B| = |B'|$, we define an *exchangeability graph* $G(B, B')$ as a bipartite graph having the vertex bipartition $(B \setminus B', B' \setminus B)$ and the edge set

$$\{(j, i) \mid j \in B \setminus B',\ i \in B' \setminus B\}.$$

Note that $|B \setminus B'| = |B' \setminus B|$ holds since B and B' have the same cardinality.

For each edge (j, i) in $G(B, B')$, we define the weight $\tilde{v}(B, j, i)$ of (j, i) by

$$\tilde{v}(B, j, i) = v(B - j + i) - v(B).$$

The value $\tilde{v}(B, j, i)$ is well defined and satisfies $\tilde{v}(B, j, i) \in \mathbb{R} \cup \{-\infty\}$ since $B \in \mathcal{F}$. It is known that the graph $G(B, B')$ has a perfect matching with a finite weight (see, e.g., [29, Cor. 39.12a]). Denote by $\widehat{v}(B, B')$ the maximum weight of a perfect matching in $G(B, B')$ with respect to the edge weight $\tilde{v}(B, j, i)$. We can bound the value $v(B')$ from above by using $v(B)$ and $\widehat{v}(B, B')$ as follows.

Proposition 4.2 (cf. [22, Lemma 3.4])**.** *For $B, B' \in \mathcal{F}$ with $|B| = |B'|$, it holds that $v(B') \leq v(B) + \widehat{v}(B, B')$.*

The statement follows from Lemma 3.4 in [22] and Proposition 2.2; a rigorous proof of this proposition is given in Appendix.

We have $|B_n| = |B^*| = m$ since $\mathcal{F}$ is a family of matroid independent sets and v is monotone nondecreasing. Hence, the following inequality follows immediately from Proposition 4.2.

Lemma 4.3. $v(B^*) \leq v(B_n) + \sum_{i \in B^* \setminus B_n} \max\{\tilde{v}(B_n, j, i) \mid j \in B_n\}$.

To bound the value $\max\{\tilde{v}(B_n, j, i) \mid j \in B_n\}$ in Lemma 4.3, we show useful inequalities for the value $\tilde{v}(B_k, j, i)$; proofs are given in Section 4.3. Let $N_k = \{i_1, i_2, \ldots, i_k\}$ for $k = 1, 2, \ldots, n$. In the following, we use the convention $-\infty \leq -\infty$.

Lemma 4.4. *For $k = 2, 3, \ldots, n$, if $B_k = B_{k-1} + i_k$, then it holds that*

$$\tilde{v}(B_k, j, i) \leq \tilde{v}(B_{k-1}, j, i) \qquad (\forall j \in B_k,\ \forall i \in N_k \setminus B_k).$$

Lemma 4.5. *For $k = 2, 3, \ldots, n$, if $B_k = B_{k-1} - j_k + i_k$, then it holds that*

$$\tilde{v}(B_k, j, j_k) \leq 0 \qquad (\forall j \in B_k), \tag{4.1}$$

$$\tilde{v}(B_k, i_k, i) \leq \tilde{v}(B_{k-1}, j_k, i) \qquad (\forall i \in N_k \setminus (B_k \cup \{j_k\})), \tag{4.2}$$

$$\tilde{v}(B_k, j, i) \leq \max\{\tilde{v}(B_{k-1}, j, i), \tilde{v}(B_{k-1}, j_k, i)\} \qquad (\forall j \in B_k \setminus \{i_k\},\ \forall i \in N_k \setminus (B_k \cup \{j_k\})). \tag{4.3}$$

Lemma 4.6. *For $k = 2, 3, \ldots, n$, if $B_k = B_{k-1}$, then it holds that*

$$\tilde{v}(B_k, j, i) = \tilde{v}(B_{k-1}, j, i) \qquad (\forall j \in B_k,\ \forall i \in N_k \setminus (B_k \cup \{i_k\})), \tag{4.4}$$

$$\tilde{v}(B_k, j, i_k) \leq \psi(t_{k-1} + 1) - \psi(t_{k-1}) \qquad (\forall j \in B_k). \tag{4.5}$$

From the three lemmas above, we can obtain a bound for the value $\max\{\tilde{v}(B_n, j, i) \mid j \in B_n\}$ as follows.

Lemma 4.7. *For $k = 2, 3, \ldots, n$, it holds that*

$$\tilde{v}(B_k, j, i) \leq \psi(t_{k-1} + 1) - \psi(t_{k-1}) \quad (\forall j \in B_k,\ \forall i \in N_k \setminus B_k). \tag{4.6}$$

Proof. We prove the claim by induction on k.

Suppose that $k = 2$. We may assume $B_2 \neq \emptyset$ and $N_2 \setminus B_2 \neq \emptyset$ since otherwise the inequality (4.6) is trivial. Then, we have at least one of the following three cases:

(a) $B_1 = \emptyset, B_2 = \{i_2\}$, (b) $B_1 = \{i_1\}, B_2 = \{i_2\}$, (c) $B_1 = B_2 = \{i_1\}$.

In each case, $j \in B_2$ and $i \in N_2 \setminus B_2$ are uniquely determined. We will prove the inequality $\tilde{v}(B_2, i_2, i_1) \leq \psi(t_1 + 1) - \psi(t_1)$ for the cases (a) and (b) and $\tilde{v}(B_2, i_1, i_2) \leq \psi(t_1 + 1) - \psi(t_1)$ for the case (c).

If (a) holds, then we have $B_2 - i_2 + i_1 = \{i_1\} \notin \mathcal{F}$ and therefore

$$\tilde{v}(B_2, i_2, i_1) = v(B_2 - i_2 + i_1) - v(B_2) = -\infty < \psi(t_1 + 1) - \psi(t_1).$$

If (b) holds, then we have $B_2 = B_1 - i_1 + i_2$, and therefore the inequality (4.1) in Lemma 4.5 implies

$$\tilde{v}(B_2, i_2, i_1) \leq 0 \leq \psi(t_1 + 1) - \psi(t_1).$$

Finally, if (c) holds, then the desired inequality follows immediately from the inequality (4.5) in Lemma 4.6.

We then suppose that $k \geq 3$. By the induction hypothesis, it holds that

$$\begin{aligned} \max\{\tilde{v}(B_{k-1}, j, i) \mid j \in B_{k-1}\} &\leq \psi(t_{k-2} + 1) - \psi(t_{k-2}) \\ &\leq \psi(t_{k-1} + 1) - \psi(t_{k-1}), \end{aligned} \tag{4.7}$$

where the second inequality is by the assumption that the value $\psi(t+1)-\psi(t)$ is monotone nondecreasing with respect to t. By Lemmas 4.4, 4.5, and 4.6, we have

$$\begin{aligned} \tilde{v}(B_k, j, i) &\leq \max\left[\max\{\tilde{v}(B_{k-1}, j, i) \mid j \in B_{k-1}\}, 0, \psi(t_{k-1} + 1) - \psi(t_{k-1})\right] \\ &\leq \psi(t_{k-1} + 1) - \psi(t_{k-1}), \end{aligned}$$

where the second inequality is by (4.7). This concludes the proof. □

Lemma 4.1 follows immediately from Lemmas 4.3 and 4.7.

4.2. Analysis of Competitive Ratio

We now prove that our online algorithm achieves the competitive ratio $r^*(\ell, c)$ in (1.1) by setting the values $\psi(t)$ $(t = 1, 2, \ldots)$ appropriately.

We consider the set of intervals given by values $\psi(t)$, and denote the length of the t-th interval as $\lambda(t) = \psi(t + 1) - \psi(t)$. Note that whenever an element is canceled in an iteration of our algorithm, the value $v(B_k) - \ell|B_k|$ moves to some upper interval. Let t_n be the integer with $v(B_n) - \ell m \in [\psi(t_n), \psi(t_n + 1))$. Then, our algorithm cancels at most $t_n - 1$ elements, and therefore the payoff obtained by the algorithm is at least $v(B_n) - (t_n - 1)c$.

By this fact and Lemma 4.1, the competitive ratio of the algorithm is at most

$$\begin{aligned}
\frac{v(B^*)}{v(B_n)-(t_n-1)c} &\le \frac{v(B_n)+m\lambda(t_{n-1})}{v(B_n)-(t_n-1)c} \\
&\le \frac{v(B_n)+m\lambda(t_n)}{v(B_n)-(t_n-1)c} \\
&\le \frac{(\psi(t_n)+\ell m)+m\lambda(t_n)}{(\psi(t_n)+\ell m)-(t_n-1)c} \\
&\le \max_{t\ge 1}\frac{(\psi(t)+\ell m)+m\lambda(t)}{(\psi(t)+\ell m)-(t-1)c},
\end{aligned} \tag{4.8}$$

where the third inequality follows from the inequality $\psi(t_n)+\ell m \le v(B_n)$ and the fact that for $p, q \in \mathbb{R}_+$ the function $(x+p)/(x-q)$ in x is decreasing in the interval $(q, +\infty)$.

Let

$$r = \max_{t\ge 1}\frac{(\psi(t)+\ell m)+m\lambda(t)}{(\psi(t)+\ell m)-(t-1)c}.$$

Then, $r \ge 1$ holds. We set values $\psi(t)$ so that $r > 1$ and

$$\frac{(\psi(t)+\ell m)+m\lambda(t)}{(\psi(t)+\ell m)-(t-1)c} = \frac{(\psi(t)+\ell m)+m(\psi(t+1)-\psi(t))}{(\psi(t)+\ell m)-(t-1)c} = r$$

for all $t \ge 1$. This implies the following recursive formula for $\psi(t)$:

$$\psi(1)=0, \qquad \psi(t+1) = \frac{m-1+r}{m}(\psi(t)+\ell m) - \frac{cr}{m}(t-1) - \ell m. \tag{4.9}$$

From the recursive formula (4.9) for $\psi(t)$, we obtain a recursive formula for $\lambda(t)$:

$$\lambda(1) = (r-1)\ell, \quad \lambda(t+1) = \alpha\lambda(t) - \beta, \quad \text{where } \alpha = \frac{m-1+r}{m},\ \beta = \frac{cr}{m},$$

and its solution is given by

$$\lambda(t) = (\lambda(1)-\gamma)\alpha^{t-1} + \gamma, \qquad \text{where } \gamma = \frac{\beta}{\alpha-1} = \frac{cr}{r-1}.$$

Since $\lambda(t)$ is monotone nondecreasing with respect to t, it holds that

$$0 \le \lambda(t+1) - \lambda(t) = (\lambda(1)-\gamma)\alpha^{t-1}(\alpha-1). \tag{4.10}$$

We have $\alpha > 1$ if $r > 1$. Therefore, (4.10) implies that

$$0 \leq \lambda(1) - \gamma = (r-1)\ell - \frac{cr}{r-1}.$$

If $r > 1$, then this inequality is equivalent to the following inequality:

$$r \geq 1 + \frac{c + \sqrt{c^2 + 4\ell c}}{2\ell} = r^*(\ell, c);$$

recall the definition of $r^*(\ell, c)$ in (1.1). Hence, we set $r = r^*(\ell, c)$, so that the competitive ratio of our algorithm is $r^*(\ell, c)$.

4.3. Proofs

4.3.1. Proof of Lemma 4.4

To prove Lemma 4.4, we use the following property.

Lemma 4.8.
(i) *For $k = 1, 2, \ldots, n$, if the element i_k is rejected in the k-th iteration, then we have $B_{k-1} + i_k \notin \mathcal{F}$.*
(ii) *For $k = 1, 2, \ldots, n$, if an element $j \in B_{k-1}$ is canceled in the k-th iteration, then we have $B_k + j \notin \mathcal{F}$.*
(iii) *For $k = 0, 1, \ldots, n-1$, if $B_k + j \notin \mathcal{F}$ holds for some $j \in N \setminus B_k$, then we have $B_{k+1} + j \notin \mathcal{F}$.*

Proof. [Proof of (i)] Since i_k is rejected in the k-th iteration of the algorithm, Case 2 occurs in this iteration and therefore $B_{k-1} + i_k \notin \mathcal{F}$ holds.

[Proof of (ii)] Since the element j is canceled in the k-th iteration of the algorithm, Case 2 occurs in this iteration, and we have $B_{k-1} + i_k \notin \mathcal{F}$ and $B_k = B_{k-1} - j + i_k$. Therefore, $B_k + j = B_{k-1} + i_k \notin \mathcal{F}$ holds.

[Proof of (iii)] If $B_{k+1} \supseteq B_k$, then we have $B_{k+1} + j \notin \mathcal{F}$ since $B_k + j \notin \mathcal{F}$ and $\mathcal{F}$ is the family of independent sets of a matroid and therefore satisfies the hereditary property. Hence, we assume $B_{k+1} \not\supseteq B_k$ in the following. Then, we must have Case 2 in the $(k+1)$-st iteration and therefore

$$B_k + i_{k+1} \notin \mathcal{F}, \quad B_{k+1} = B_k - j_{k+1} + i_{k+1}.$$

Assume, to the contrary, that $B_{k+1} + j \in \mathcal{F}$. We have

$$B_k \in \mathcal{F}, \quad |B_k| < |B_{k+1} + j|, \quad (B_{k+1} + j) \setminus B_k = \{j, i_{k+1}\},$$

which, together with the augmentation property of the matroid $\mathcal{F}$, implies that at least one of $B_k + j \in \mathcal{F}$ and $B_k + i_{k+1} \in \mathcal{F}$, a contradiction. Hence, we have $B_{k+1} + j \notin \mathcal{F}$. $\square$

Proof of Lemma 4.4. Put $B = B_k - j + i$. If $B \notin \mathcal{F}$, then we have

$$\tilde{v}(B_k, j, i) = -\infty < \tilde{v}(B_{k-1}, j, i).$$

Hence, we assume $B \in \mathcal{F}$ in the following.

Lemma 4.8 implies that $B_{k-1}+i \notin \mathcal{F}$ since $i \in N_k \setminus B_k$ is an element that is rejected or canceled in a previous iteration of the algorithm. If $j = i_k$, then we have $B = B_{k-1} + i \notin \mathcal{F}$, a contradiction to the assumption $B \in \mathcal{F}$. Hence, we have $j \in B_k \setminus \{i_k\} = B_{k-1}$, implying $B = B_{k-1} \setminus \{j\} \cup \{i, i_k\}$. Since $B_{k-1}+i \notin \mathcal{F}$, (M$^\natural$-EXC) applied to B, B_{k-1}, and $i \in B \setminus B_{k-1}$ implies that

$$\begin{aligned}
&B - i + j = B_k \in \mathcal{F}, \quad B_{k-1} + i - j \in \mathcal{F}, \\
&v(B) + v(B_{k-1}) \leq v(B_k) + v(B_{k-1} + i - j).
\end{aligned}$$

It follows that

$$\tilde{v}(B_k, j, i) = v(B) - v(B_k) \leq v(B_{k-1} + i - j) - v(B_{k-1}) = \tilde{v}(B_{k-1}, j, i).$$

$\square$

4.3.2. Proof of Lemma 4.5

We first prove the inequality (4.1). If $j = i_k$, then

$$\tilde{v}(B_k, i_k, j_k) = v(B_{k-1}) - v(B_k) \leq 0.$$

If $j \in B_k \setminus \{i_k\}$, then

$$\begin{aligned}
\tilde{v}(B_k, j, j_k) &= v(B_k - j + j_k) - v(B_k) \\
&= v(B_{k-1} - j + i_k) - v(B_{k-1} - j_k + i_k) \\
&= \tilde{v}(B_{k-1}, j, i_k) - \tilde{v}(B_{k-1}, j_k, i_k) \leq 0,
\end{aligned}$$

where the inequality follows from the choice of j_k since $B_k \setminus \{i_k\} \subseteq B_{k-1}$. Hence, (4.1) follows.

We next prove the inequality (4.2). It holds that

$$\begin{aligned}
\tilde{v}(B_k, i_k, i) &= v(B_k - i_k + i) - v(B_k) \\
&= v(B_{k-1} - j_k + i) - v(B_{k-1} - j_k + i_k) \\
&= \tilde{v}(B_{k-1}, j_k, i) - \tilde{v}(B_{k-1}, j_k, i_k) < \tilde{v}(B_{k-1}, j_k, i),
\end{aligned}$$

where the inequality follows from the fact that j_k is canceled in the k-th iteration. Hence, (4.2) holds.

We finally prove (4.3). Let

$$B = B_k - j + i \ (= B_{k-1} \setminus \{j, j_k\} \cup \{i, i_k\}).$$

Since $|B| = |B_{k-1}|$, Proposition 2.2 applied to B and B_{k-1} implies that

$$\begin{aligned}
&v(B) + v(B_{k-1}) \\
&\leq \max\{v(B + j - i) + v(B_{k-1} - j + i), v(B + j - i_k) + v(B_{k-1} - j + i_k)\} \\
&= \max\{v(B_{k-1} - j_k + i_k) + v(B_{k-1} - j + i), \\
&\qquad\qquad\qquad v(B_{k-1} - j_k + i) + v(B_{k-1} - j + i_k)\},
\end{aligned}$$

from which follows that

$$\begin{aligned}
&v(B) - v(B_{k-1}) \\
&\leq \max\{\tilde{v}(B_{k-1}, j_k, i_k) + \tilde{v}(B_{k-1}, j, i), \tilde{v}(B_{k-1}, j_k, i) + \tilde{v}(B_{k-1}, j, i_k)\}.
\end{aligned}$$

Hence, it holds that

$$\begin{aligned}
&\tilde{v}(B_k, j, i) = v(B) - v(B_k) \\
&\leq \max\{\tilde{v}(B_{k-1}, j_k, i_k) + \tilde{v}(B_{k-1}, j, i), \tilde{v}(B_{k-1}, j_k, i) + \tilde{v}(B_{k-1}, j, i_k)\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + v(B_{k-1}) - v(B_k) \\
&= \max\{\tilde{v}(B_{k-1}, j_k, i_k) + \tilde{v}(B_{k-1}, j, i), \tilde{v}(B_{k-1}, j_k, i) + \tilde{v}(B_{k-1}, j, i_k)\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \tilde{v}(B_{k-1}, j_k, i_k) \\
&= \max\{\tilde{v}(B_{k-1}, j, i), \tilde{v}(B_{k-1}, j_k, i) + \tilde{v}(B_{k-1}, j, i_k) - \tilde{v}(B_{k-1}, j_k, i_k)\} \\
&\leq \max\{\tilde{v}(B_{k-1}, j, i), \tilde{v}(B_{k-1}, j_k, i)\},
\end{aligned}$$

where the last inequality follows from the choice of j_k since $B_k \setminus \{i_k\} \subseteq B_{k-1}$. Hence, (4.3) holds.

4.3.3. Proof of Lemma 4.6

Since $B_k = B_{k-1}$, (4.4) holds trivially. To prove (4.5), let $j \in B_k$. Since i_k is rejected, we have

$$\psi(t_{k-1} + 1) + \ell \cdot |B_{k-1}| > v(B_{k-1} - j + i_k).$$

Hence, it holds that

$$\begin{aligned}
\tilde{v}(B_k, j, i_k) &= v(B_k - j + i_k) - v(B_k) \\
&= v(B_{k-1} - j + i_k) - v(B_{k-1}) \\
&< (\psi(t_{k-1} + 1) + \ell \cdot |B_{k-1}|) - (\psi(t_{k-1}) + \ell \cdot |B_{k-1}|) \\
&= \psi(t_{k-1} + 1) - \psi(t_{k-1}),
\end{aligned}$$

where we use the fact that $v(B_{k-1}) - \ell \cdot |B_{k-1}| \geq \psi(t_{k-1})$.

5. Concluding Remarks

We have shown that the competitive ratio of our online algorithm for M$^\natural$-concave buyback problem is $r^*(\ell, c)$. Note that $\lim_{c \downarrow 0} r^*(\ell, c) = r^*(\ell, 0) = 1$, which means that our online algorithm finds an offline optimal solution if c is a sufficiently small positive real number. Indeed, if c is sufficiently small, then the values $\psi(t+1) - \psi(t)$ are also sufficiently small for all t, implying that the behavior of the algorithm M$^\natural$BP with a sufficiently small c coincides with that of the algorithm M$^\natural$Greedy.

In this paper, we consider the setting where the cancellation cost is the same for all items. We may consider the setting where the cancellation cost is non-uniform and c_i denotes the cancellation cost of item $i \in N$. The problem with non-uniform costs includes as a special case the linear buyback problem with proportional cancellation costs discussed in [3, 4, 10]. If the cancellation costs c_i $(i \in N)$ are known in advance before starting the algorithm, then the same online algorithm with $c = \max_{i \in N} c_i$ achieves the competitive ratio $r^*(\ell, c)$. On the other hand, if the cancellation costs are not known in advance, then we do not know any non-trivial bound of the competitive ratio.

To obtain our main result (i.e., Theorem 1.3), we have assumed that the value ℓ with (1.3) is known in advance. In fact, even if such ℓ is not known in advance, we can obtain the same competitive ratio by a slight modification of the proposed algorithm. The idea is to use the value ℓ_k given by

$$\ell_1 = 0, \quad \ell_k = \min\{v(B_i)/|B_i| \mid 1 \leq i \leq k-1\} \ (k = 2, 3, \ldots, n)$$

in the k-th iteration instead of the original ℓ. By definition, we have

$$\begin{aligned} &\ell_1 \geq \ell_2 \geq \cdots \geq \ell_n \geq \min\{v(X)/|X| \mid \emptyset \neq X \in \mathcal{F}\}, \\ &0 \ \leq v(B_1) - \ell_k \cdot |B_1| \leq v(B_2) - \ell_k \cdot |B_2| \leq \cdots \leq v(B_k) - \ell_k \cdot |B_k| \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (k = 1, 2, \ldots, n). \end{aligned}$$

Therefore, we can show that if $v(B_n) - \ell_n m \in [\psi(t), \psi(t+1))$ for some integer t, then the cancellation occurs at most $t-1$ times. Hence, we can obtain the same competitive ratio $r^*(\ell, c)$ with $\ell = \min\{v(X)/|X| \mid \emptyset \neq X \in \mathcal{F}\}$ in a similar way as in Section 4.2.

Finally, we note that our approach does not extend to the nonlinear buyback problem with a general submodular valuation function. To illustrate this, let us consider an instance of the buyback problem such that

$N = \{i_1, i_2, i_3, i_4\}$, the valuation function $v : 2^N \to \mathbb{R}$ is given by

$$\begin{aligned}
&v(\emptyset) = 0, \quad v(\{i_1\}) = v(\{i_2\}) = 2,\ v(\{i_3\}) = v(\{i_4\}) = 3,\\
&v(X) = 6 \text{ if } |X| \geq 2 \text{ and } X \supseteq \{i_3, i_4\}\\
&v(X) = 4 \text{ if } |X| = 2 \text{ and } X \neq \{i_3, i_4\},\\
&v(N \setminus \{i_3\}) = v(N \setminus \{i_4\}) = 5,
\end{aligned}$$

and the constraint is $\mathcal{F} = \{X \in 2^N \mid |X| \leq 2\}$. It can be checked that the function v is submodular but not M$^\natural$-concave.

Suppose that our online algorithm is applied to this instance, where the elements i_1, i_2, i_3, i_4 arrive in this order. Then, the algorithm first accepts elements i_1 and i_2, and then rejects i_3 and i_4 since the function value cannot be increased by swapping new elements with old elements one by one. Hence, the value of the output is $v(\{i_1, i_2\}) = 4$. Note that this behavior of the algorithm is irrelevant to the choice of the cancellation cost c. On the other hand, an offline optimal solution is $B^* = \{i_3, i_4\}$, for which $v(B^*) = 6$. Hence, the competitive ratio of our algorithm is at least $6/4 = 1.5$, while the ratio $r^*(\ell, c)$ can be close to 1 if we choose a sufficiently small positive c. This fact shows that our algorithm and analysis in this paper do not extend to submodular valuation functions.

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Appendix A. Examples of M$^\natural$-concave Functions and GS Valuation Functions

We give some nontrivial examples of M$^\natural$-concave functions and GS valuation functions.

Example 1 (Maximum-weight bipartite matching)**.** In Section 1.2 we explained an assignment valuation as an example of GS valuations, where a complete bipartite graph is used. By using a non-complete bipartite graph instead, we can obtain an example of M$^\natural$-concave functions as follows.

Consider a bipartite graph G with two vertex sets N, J and an edge set E ($\subseteq N \times J$), where N and J correspond to workers and jobs, respectively. An edge $(i, j) \in E$ means that worker $i \in N$ has ability to process job

$j \in J$, and profit $q(i,j) \in \mathbb{R}_+$ can be obtained by assigning worker i to job j. Consider a matching between workers and jobs, and define $\mathcal{F} \subseteq 2^N$ by

$$\mathcal{F} = \{X \subseteq N \mid \exists M : \text{matching in } G \text{ s.t. } \partial_N M = X\}.$$

It is well known that $\mathcal{F}$ is the family of independent sets in a transversal matroid (see, e.g., [29]). Define a function $v : 2^N \to \mathbb{R} \cup \{-\infty\}$ by

$$\begin{aligned} &\operatorname{dom} v = \mathcal{F}, \\ &v(X) = \max \Big\{ \sum_{(i,j) \in M} q(i,j) \;\Big|\; M : \text{matching in } G \text{ s.t. } \partial_N M = X \Big\} \ (X \in \operatorname{dom} v). \end{aligned}$$

Then, v is an M$^\natural$-concave function (see, e.g., [23], [26, Sec. 11.4.2]). □

Example 2 (Laminar concave functions)**.** Let $\mathcal{T} \subseteq 2^N$ be a laminar family, i.e., $X \cap Y = \emptyset$ or $X \subseteq Y$ or $X \supseteq Y$ holds for every $X, Y \in \mathcal{T}$. For $Y \in \mathcal{T}$, let $\varphi_Y : \mathbb{Z}_+ \to \mathbb{R}$ be a univariate concave function. Define a function $v : 2^N \to \mathbb{R}$ by

$$v(X) = \sum_{Y \in \mathcal{T}} \varphi_Y(|X \cap Y|) \quad (X \in 2^N),$$

which is called a *laminar concave function* [25, Sec. 6.3] (also called an *S-valuation* in [5]). Special cases of laminar concave functions are a *downward sloping symmetric function* [11] given as $v(X) = \varphi(|X|)$ and a *nested concave function* given as

$$v(X) = \sum_{i=1}^{n} \varphi_i(|X \cap \{1, 2, \ldots, i\}|),$$

where φ and φ_i ($i \in N$) are univariate concave functions. Every laminar concave function is a GS valuation function. □

Example 3 (Weighted rank functions)**.** Let $\mathcal{I} \subseteq 2^N$ be the family of independent sets of a matroid, and $w \in \mathbb{R}_+^N$. Define a function $v : 2^N \to \mathbb{R}_+$ by

$$v(X) = \max\{w(Y) \mid Y \subseteq X,\ Y \in \mathcal{I}\} \quad (X \in 2^N),$$

which is called the *weighted rank function* [9]. If $w(i) = 1$ ($i \in N$), then v is an ordinary rank function of the matroid $(N, \mathcal{I})$. Every weighted rank function is a GS valuation function [32]. □

Appendix B. Proof of Proposition 3.1

The validity of the algorithm M$^\natural$Greedy can be shown by using the following properties of an M$^\natural$-concave function.

Proposition B.1 (cf. [31, Th. 2.1], [25, Th. 6.28]). *Let $v : 2^N \to \mathbb{R} \cup \{-\infty\}$ be an M$^\natural$-concave function. Let $X \in \mathrm{dom}\, v$ and $i \in N \setminus X$ be such that $X + i \notin \mathrm{dom}\, v$. If $h \in X + i$ satisfies*

$$v(X - h + i) = \max\{v(X - j + i) \mid h \in X + i\},$$

then there exists a maximizer $X^* \subseteq N$ of function v such that $h \notin X^*$.

Proof. Let Y^* be a maximizer of function v. If $h \notin Y^*$, then we are done. Hence, we assume $h \in Y^*$ in the following, and show that there exists another maximizer of v not containing h.

Putting $Y = X - h + i$, we have $h \in Y^* \setminus Y$. Therefore, (M$^\natural$-EXC) implies that we have either

$$v(Y^*) + v(Y) \leq v(Y^* - h) + v(Y + h)$$

or

$$v(Y^*) + v(Y) \leq v(Y^* - h + j) + v(Y + h - j) \tag{B.1}$$

for some $j \in Y \setminus Y^*$ (or both). Since $Y + h = X + i \notin \mathrm{dom}\, v$, we have $v(Y + h) = -\infty$, implying that the inequality (B.1) holds. Since $j \in Y \subseteq (X + i)$, we have

$$v(Y) = v(X - h + i) \geq v(X + i - j) = v(Y + h - j),$$

which, together with (B.1), implies that $v(Y^*) \leq v(Y^* - h + j)$, i.e., $Y^* - h + j$ is also a maximizer of function v not containing h. □

We now give a proof of Proposition 3.1. We claim that

> **(P)** for each $k = 0, 1, \ldots, n$, there exists a maximizer B_k^* of function v such that $B_k^* \subseteq M_k$ and $|B_k^*| \geq |B_k|$,

where $M_k = B_k \cup \{i_{k+1}, i_{k+2}, \ldots, i_n\}$. Note that (P) implies $B_n^* = B_n$, and therefore the output B_n of the algorithm is a maximizer of v. In the following we prove (P) by induction on k. The property (P) holds trivially if $k = 0$. Hence, we assume $k > 0$.

We first consider the case with $B_k = B_{k-1} + i_k$. Then, we have $M_k = M_{k-1}$. By the induction hypothesis, there exists a maximizer B_{k-1}^* of function v such that $B_{k-1}^* \subseteq M_{k-1}$ and $|B_{k-1}^*| \geq |B_{k-1}|$. Since $B_k \in \mathcal{F}$ and

$B_k \subseteq M_k$, repeated application of the augmentation property of the matroid $\mathcal{F}$ implies that there exists some $B^* \in \mathcal{F}$ such that $B^*_{k-1} \subseteq B^* \subseteq M_k$ and $|B^*| \geq |B_k|$. By the monotonicity of function v, we have $v(B^*) \geq v(B^*_{k-1})$, i.e., B^* is also a maximizer of v. Hence, (P) holds by setting $B^*_k = B^*$.

We then consider the case with $B_k = B_{k-1} - j_k + i_k$ or $B_k = B_{k-1}$. In this case, we have $B_{k-1} + i_k \notin \mathcal{F}$ by the behavior of the algorithm. Denoting

$$h = \begin{cases} j_k & (\text{if } B_k = B_{k-1} - j_k + i_k), \\ i_k & (\text{if } B_k = B_{k-1}), \end{cases}$$

we have $B_k = B_{k-1} - h + i_k$.

Define a function $v_{k-1} : 2^{M_{k-1}} \to \mathbb{R} \cup \{-\infty\}$ by

$$v_{k-1}(X) = \begin{cases} v(X) & (\text{if } |X| \geq |B_{k-1}|), \\ -\infty & (\text{otherwise}) \end{cases} \qquad (X \subseteq M_{k-1}).$$

Note that v_{k-1} is an $\mathrm{M}^\natural$-concave function. The induction hypothesis implies that

$$\max\{v_{k-1}(X) \mid X \subseteq M_{k-1}\} = \max\{v(X) \mid X \subseteq N\}. \tag{B.2}$$

We also have

$$v_{k-1}(B_k) = \max\{v_{k-1}(B_{k-1} + i_k - j) \mid j \in B_{k-1} + i_k\}$$

by the definition of B_k. This equation, together with Proposition B.1 applied to v_{k-1}, implies that there exists a maximizer $B^{**} \subseteq M_{k-1}$ of function v_{k-1} such that $h \notin B^{**}$. Hence, we have $B^{**} \subseteq M_{k-1} - h = M_k$. Since $B^{**} \in \mathrm{dom}\, v_{k-1}$, we have $|B^{**}| \geq |B_{k-1}| = |B_k|$ by the definition of v_{k-1}. Moreover, B^{**} is a maximizer of v by (B.2). Therefore, (P) holds. This concludes the proof of Proposition 3.1.

Appendix C. Proof of Proposition 4.2

We prove the inequality by induction on the cardinality of $B \setminus B'$. If $|B \setminus B'| = 0$, then we have $B = B'$ and $\widehat{v}(B, B') = 0$ since $G(B, B')$ has no vertex. Hence, $v(B') = v(B) + \widehat{v}(B, B')$ holds.

We then assume $|B \setminus B'| > 0$ and let $j \in B \setminus B'$. Then, Proposition 2.2 applied to B, B', and j implies that there exists some $i \in B' \setminus B$ such that

$$v(B) + v(B') \leq v(B - j + i) + v(B' + j - i).$$

We denote $B'' = B' + j - i$. Then, we have

$$v(B') \leq v(B'') + \tilde{v}(B, j, i).$$

Since $|B \setminus B''| < |B \setminus B'|$, we can apply the induction hypothesis to B and B'' to obtain

$$v(B'') \leq v(B) + \widehat{v}(B, B'').$$

Hence, it follows that

$$\begin{aligned} v(B') \leq v(B'') + \tilde{v}(B, j, i) &\leq v(B) + \widehat{v}(B, B'') + \tilde{v}(B, j, i) \\ &\leq v(B) + \widehat{v}(B, B'), \end{aligned}$$

where the last inequality follows from the fact that edges in a perfect matching in $G(B, B'')$, together with the edge (j, i), gives a perfect matching in $G(B, B')$.

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