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Relationship of M-/L-convex Functions with Discrete Convex Functions by Miller and by Favati–Tardella

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Abstract

We clarify the relationship of the concepts of M-convex and L-convex functions due to Murota (1996, 1998) with two other concepts of discrete convex functions over integer lattice points, discretely-convex functions due to Miller (1971), and integrally-convex functions due to Favati–Tardella (1990). We also investigate whether each class of discrete convex functions is closed under fundamental operations such as addition and convolution.

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1 Introduction

The convexity concept for sets and functions plays a pivotal role in the area of continuous optimization (or nonlinear optimization with continuous variables) [7, 14, 15]. One of the most important properties of convex functions is that the local optimality guarantees the global optimality. This property allows us to find the minimum of a convex function by iteratively moving in descent directions. Namely, the so-called "greedy algorithms" work for convex functions.

In discrete optimization, on the other hand, discrete analogues of convexity, or "discrete convexity" for short, have been considered, with a view to identifying the discrete structure that guarantees the success of greedy algorithms. Consequently, several different types of discrete convexity have been proposed.

Miller [8] investigated a class of discrete functions, called "discretely-convex" functions, such that local optimality implies global optimality (see Theorem 2.2). Favati–Tardella [1] considered a certain special way of extending functions defined over the integer lattice to piecewise-linear functions defined over the real space, and introduced the concept of "integrally-convex" functions.

The concepts of "M-convexity" and "L-convexity", introduced by Murota [9, 10, 11], afford a nice framework for discrete optimization problems. M-convex/L-convex functions have various desirable properties as discrete convex functions: extendibility to ordinary convex functions, duality theorems, conjugacy between M/L-convex functions, etc.

Variants of M-convex and L-convex functions, called "$\mathrm{M}^{\natural}$-convex" and "$\mathrm{L}^{\natural}$-convex" functions, are introduced by Murota–Shioura [13] and Fujishige–Murota [6], respectively. $\mathrm{M}^{\natural}$-convex (resp. $\mathrm{L}^{\natural}$-convex) functions are essentially equivalent to M-convex (resp. L-convex) functions, whereas the class of $\mathrm{M}^{\natural}$-convex (resp. $\mathrm{L}^{\natural}$-convex) functions properly contains that of M-convex (resp. L-convex) functions. It is shown in [6] that the class of $\mathrm{L}^{\natural}$-convex functions coincides with that of submodular integrally-convex functions considered in [1].

In this paper, we clarify the relationship of M-convexity/L-convexity with discrete convexity by Miller and by Favati–Tardella. Miller's discrete convexity contains the other classes of discrete convexity (Theorem 3.3), M-convexity/L-convexity are special cases of discrete convexity by Favati–Tardella (Theorems 3.9, 3.12), and the class of separable-convex functions coincides with the intersection of the classes of $\mathrm{M}^{\natural}$-convex/$\mathrm{L}^{\natural}$-convex functions (Theorem 3.17). We also discuss some fundamental operations for discrete convex functions, such as addition, convolution, and the Fenchel-Legendre transformation. We check whether each discrete convexity is closed under each operation, and provide a proof or a counterexample for the statement.

The organization of this paper is as follows. Section 2 explains notation and provides the definitions of discrete convexity. We then show the relationship between various discrete convexity in Section 3, and discuss the operations for discrete convexity in Section 4. Section 5 provides proofs of some theorems.

2 Definitions on Discrete Convex Functions

We give the definitions of discretely-convex, integrally-convex, M-convex, and L-convex functions.

We denote by $\mathbf{R}$ the set of reals, and by $\mathbf{Z}$ the set of integers. Let V be a nonempty finite set. The characteristic vector of a subset $X \subseteq V$ is denoted by χ_X ($\in \{0,1\}^V$), i.e.,

$$\chi_X(w) = \begin{cases} 1 & (w \in X), \\ 0 & (w \in V - X). \end{cases}$$

In particular, we use the notation $\mathbf{0} = \chi_\emptyset$ and $\mathbf{1} = \chi_V$.

For $a : V \to \mathbf{R} \cup \{-\infty\}$ and $b : V \to \mathbf{R} \cup \{+\infty\}$ with $a(v) \le b(v)$ ($v \in V$), we define the interval $[a,b]$ ($\subseteq \mathbf{R}^V$) by

$$[a,b] = \{x \in \mathbf{R}^V \mid a \le x \le b\}.$$

For $x \in \mathbf{R}^V$, we define the sets

$$\begin{aligned} \mathrm{N}_0(x) &= \{y \in \mathbf{Z}^V \mid \lfloor x \rfloor \le y \le \lceil x \rceil\}, \\ \mathrm{N}_1(x) &= \{y \in \mathbf{Z}^V \mid \lfloor x \rfloor - \mathbf{1} \le y \le \lceil x \rceil + \mathbf{1}\}, \end{aligned}$$

where $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) denotes the vector obtained by rounding down (resp. up) the components of x to the nearest integers. In particular, $\mathrm{N}_0(x)$ denotes the set of integral vectors in the smallest hypercube containing x.

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. We define $\mathrm{dom}\, f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\}$. A function f is said to be *discretely-convex* if for any $x', x'' \in \mathrm{dom}\, f$ and any $\alpha \in [0,1]$, it holds that

$$\min\{f(y) \mid y \in \mathrm{N}_0(\alpha x' + (1-\alpha)x'')\} \le \alpha f(x') + (1-\alpha) f(x'').$$

Remark 2.1 The definition of discretely-convex functions in this paper is slightly different from the original one by Miller [8], where it is defined for functions over the set of integral vectors in a closed interval. Our definition is based on the "weaker requirement" in [8]. □

The local optimality implies the global optimality for discretely-convex functions.

Theorem 2.2 ([8]) *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be discretely-convex and $x \in \mathrm{dom}\, f$. Then, $f(x) \le f(y)$ for all $y \in \mathbf{Z}^V$ if and only if $f(x) \le f(y)$ for all $y \in \mathrm{N}_1(x)$.*

We also introduce discrete convexity for sets. For any $S \subseteq \mathbf{Z}^V$, its *indicator function* $\delta_S : \mathbf{Z}^V \to \{0, +\infty\}$ is defined as

$$\delta_S(x) = \begin{cases} 0 & (x \in S), \\ +\infty & (x \notin S). \end{cases}$$

A set $S \subseteq \mathbf{Z}^V$ is called a *discretely-convex set* if δ_S is a discretely-convex function. Alternatively, a set $S \subseteq \mathbf{Z}^V$ is discretely-convex if for any $x', x'' \in S$ and any $\alpha \in [0,1]$, it holds that $\mathrm{N}_0(\alpha x' + (1-\alpha)x'') \cap S \ne \emptyset$. In this paper, we do not distinguish a set of integral vectors and its indicator function, and when a concept of "convex" functions is given, we call a set $S \subseteq \mathbf{Z}^V$ "convex" if its indicator function $\delta_S : \mathbf{Z}^V \to \{0, +\infty\}$ is a "convex" function.

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. A closed convex function $f_{\mathbf{R}} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is called a *convex extension* of f if $f_{\mathbf{R}}(x) = f(x)$ for all $x \in \mathbf{Z}^V$. The *convex closure* $\overline{f} : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ of f is defined by

$$\overline{f}(x) = \sup_{p \in \mathbf{R}^V, \alpha \in \mathbf{R}} \{\langle p, x\rangle + \alpha \mid \langle p, y\rangle + \alpha \leq f(y) \ (\forall y \in \mathbf{Z}^V)\} \qquad (x \in \mathbf{R}^V), \tag{2.1}$$

where $\langle p, y\rangle = \sum_{v \in V} p(v)y(v)$. For $S \subseteq \mathbf{R}^V$, the *convex closure* of S, denoted by $\overline{S}$, is the smallest closed convex set containing S.

Lemma 2.3 *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function. Then, $\overline{f}(x) = f(x)$ for any $x \in \mathbf{Z}^V$ if and only if there exists a convex extension of f.*

We call a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *convex-extendible* if $\overline{f}(x) = f(x)$ for any $x \in \mathbf{Z}^V$. A set $S \subseteq \mathbf{Z}^V$ is said to be *convex-extendible* if $\overline{S} \cap \mathbf{Z}^V = S$.

We next introduce the *local convex extension* of a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. Define $\tilde{f} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$\tilde{f}(x) = \sup_{p \in \mathbf{R}^V, \alpha \in \mathbf{R}} \{\langle p, x\rangle + \alpha \mid \langle p, y\rangle + \alpha \leq f(y) \ (\forall y \in \mathrm{N}_0(x))\} \qquad (x \in \mathbf{R}^V). \tag{2.2}$$

Note that $\tilde{f}$ is the convex closure of the restriction of f to the integral points around x. It admits an alternative expression

$$\tilde{f}(x) = \inf\{\sum_{y \in \mathrm{N}_0(x)} \lambda_y f(y) \mid \sum_{y \in \mathrm{N}_0(x)} \lambda_y y = x, \ \sum_{y \in \mathrm{N}_0(x)} \lambda_y = 1, \ \lambda_y \geq 0 \ (y \in \mathrm{N}_0(x))\} \quad (x \in \mathbf{R}^V) \tag{2.3}$$

by the linear programming duality. From the definitions, we have

$$\tilde{f}(x) \geq \overline{f}(x) \quad (\forall x \in \mathbf{R}^V), \qquad \tilde{f}(x) = f(x) \quad (\forall x \in \mathbf{Z}^V). \tag{2.4}$$

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be *integrally-convex* if its local convex extension $\tilde{f} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is a convex function, or equivalently, if $\tilde{f} = \overline{f}$. A set $S \subseteq \mathbf{Z}^V$ is *integrally-convex* if

$$\overline{S \cap \mathrm{N}_0(x)} = \overline{S} \cap \overline{\mathrm{N}_0(x)} \qquad (\forall x \in \mathbf{R}^V). \tag{2.5}$$

We say a function $f : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$ is *convex* if $f(\alpha - 1) + f(\alpha + 1) \geq 2f(\alpha)$ for any $\alpha \in \mathbf{Z}$. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be *separable-convex* if $f(x) = \sum_{v \in V} f_v(x(v))$ $(x \in \mathbf{Z}^V)$ for a family of convex functions $f_v : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$ $(v \in V)$. Note that a *separable-convex set* is nothing but the set of integral vectors in the interval $[a, b]$ for some $a : V \to \mathbf{R} \cup \{-\infty\}$ and $b : V \to \mathbf{R} \cup \{+\infty\}$.

For any $x, y \in \mathbf{Z}^V$, the vectors $x \wedge y, x \vee y \in \mathbf{Z}^V$ are such that

$$(x \wedge y)(v) = \min\{x(v), y(v)\}, \quad (x \vee y)(v) = \max\{x(v), y(v)\} \quad (v \in V).$$

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be

$$\begin{array}{rcll} \textit{submodular} & \Longleftrightarrow & f(x)+f(y) \geq f(x \wedge y)+f(x \vee y) & (\forall x, y \in \mathbf{Z}^V), \\ \textit{supermodular} & \Longleftrightarrow & f(x)+f(y) \leq f(x \wedge y)+f(x \vee y) & (\forall x, y \in \mathbf{Z}^V), \\ \textit{modular} & \Longleftrightarrow & f(x)+f(y) = f(x \wedge y)+f(x \vee y) & (\forall x, y \in \mathbf{Z}^V). \end{array}$$

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called *M-convex* if it satisfies

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that

$$f(x)+f(y) \geq f(x-\chi_u+\chi_v)+f(y+\chi_u-\chi_v),$$

where

$$\mathrm{supp}^+(x-y)=\{v \in V \mid x(v)>y(v)\}, \quad \mathrm{supp}^-(x-y)=\{v \in V \mid x(v)<y(v)\}.$$

A function is said to be *M_2-convex* if it is represented as the sum of two M-convex functions. Note that an *M-convex set* is nothing but the set of integral vectors in an integral base polyhedron [5].

The effective domain of an M-convex function is contained in a hyperplane $\{x \in \mathbf{Z}^V \mid \sum_{v \in V} x(v) = r\}$ for some $r \in \mathbf{Z}$ (cf. [11, Theorem 4.3]). Therefore, no information is lost when an M-convex function is projected onto a $(|V|-1)$-dimensional integer lattice. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called *M$^\natural$-convex* if the function $f_0 : \mathbf{Z} \times \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f_0(x_0, x) = \begin{cases} f(x) & (x_0 + \sum_{v \in V} x(v) = 0), \\ +\infty & (x_0 + \sum_{v \in V} x(v) \neq 0), \end{cases} \qquad ((x_0, x) \in \mathbf{Z} \times \mathbf{Z}^V) \tag{2.6}$$

is an M-convex function. The exchange property (M-EXC) for f_0 is translated as follows [13]:

(M$^\natural$-EXC) $\forall x, y \in \mathrm{dom}_{\mathbf{Z}}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, either (i) or (ii) (or both) holds:
(i) $\exists v \in \mathrm{supp}^-(x-y)$ such that $f(x)+f(y) \geq f(x-\chi_u+\chi_v)+f(y+\chi_u-\chi_v)$,
(ii) $f(x)+f(y) \geq f(x-\chi_u)+f(y+\chi_u)$.

Therefore, a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is M$^\natural$-convex if and only if f satisfies (M$^\natural$-EXC). M$^\natural$-convex functions are essentially equivalent to M-convex functions, whereas the class of M$^\natural$-convex functions properly contains that of M-convex functions. A function is said to be *$M_2^\natural$-convex* if it is represented as the sum of two M$^\natural$-convex functions. An *M$^\natural$-convex set* is equivalent to an integral generalized polymatroid by Frank [2] (see also Frank–Tardos [3]).

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called *L-convex* if it satisfies

(LF1) f is submodular,
(LF2) $\exists r \in \mathbf{R}$ such that $f(x+\alpha \mathbf{1}) = f(x)+\alpha r \quad (\forall x \in \mathrm{dom}\, f,\ \forall \alpha \in \mathbf{Z})$.

For any two functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, the *convolution* of f_1 and f_2, denoted by $f_1 \Box f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{\pm\infty\}$, is defined by

$$(f_1 \Box f_2)(x) = \inf\{f_1(x_1)+f_2(x_2) \mid x_1, x_2 \in \mathbf{Z}^V,\ x_1+x_2=x\} \qquad (x \in \mathbf{Z}^V).$$

For any two sets $S_1, S_2 \subseteq \mathbf{Z}^V$, the *Minkowski-sum* of S_1 and S_2, denoted by $S_1 + S_2$, is defined by

$$S_1 + S_2 = \{x_1 + x_2 \mid x_1 \in S_1,\ x_2 \in S_2\}\ (\subseteq \mathbf{Z}^V).$$

A function is said to be *L_2-convex* if it is represented as the convolution of two L-convex functions. Accordingly, a set is called *L_2-convex* if it is represented as the Minkowski-sum of two L-convex sets.

Due to the property (LF2), an L-convex function loses no information when restricted to a hyperplane $\{x \in \mathbf{Z}^V \mid x(v) = 0\}$ for any $v \in V$. We call a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *$L^\natural$-convex* if the function $f^0 : \mathbf{Z} \times \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f^0(x_0, x) = f(x - x_0\mathbf{1}) \quad ((x_0, x) \in \mathbf{Z} \times \mathbf{Z}^V) \tag{2.7}$$

is L-convex. It is known [6] that L$^\natural$-convex functions are essentially the same as L-convex functions, while the class of L$^\natural$-convex functions properly contains that of L-convex functions. A function is said to be *$L_2^\natural$-convex* if it is represented as the convolution of two L$^\natural$-convex functions. An L$^\natural$-convex function can be characterized by the discrete mid-point convexity:

$$f(x) + f(x) \geq f(\left\lfloor \frac{x+y}{2} \right\rfloor) + f(\left\lceil \frac{x+y}{2} \right\rceil) \qquad (\forall x, y \in \mathbf{Z}^V). \tag{2.8}$$

Theorem 2.4 ([6]) *A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is $L^\natural$-convex if and only if f satisfies the mid-point convexity (2.8).*

Remark 2.5 The original definitions of M-convex/L-convex functions in [9, 10, 11] assume that the effective domain is nonempty. This paper removes the nonemptyness assumption for convenience. □

3 Relationship among Discrete Convex Functions

In this section, we clarify the relationship among various discrete convexity for functions defined over the integer lattice. The relationship between discrete convexity and submodularity/supermodularity is also discussed. As a special but important case, we also refer to functions defined over $\{0, 1\}$ vectors, which are equivalent to set functions $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$ under a natural correspondence between $X \subseteq V$ and $\chi_X \in \{0, 1\}^V$. The results in this section are summarized in Figure 1, which shows "M$^\natural$-convex ∩ L$^\natural$-convex = M$_2^\natural$-convex ∩ L$_2^\natural$-convex = separable-convex", in particular.

First we note that there is no inclusion relationship between the class of discretely convex functions and that of convex-extendible functions.

Example 3.1 (G. Károlyi) This is an example of a discretely-convex set which is not convex-extendible. The set

$$S = \{x \in \mathbf{Z}^3 \mid x_1 + x_2 + x_3 = 2,\ x_i \geq 0\ (i = 1, 2, 3)\} \cup \{(1, 2, 0), (0, 1, 2), (2, 0, 1)\}$$

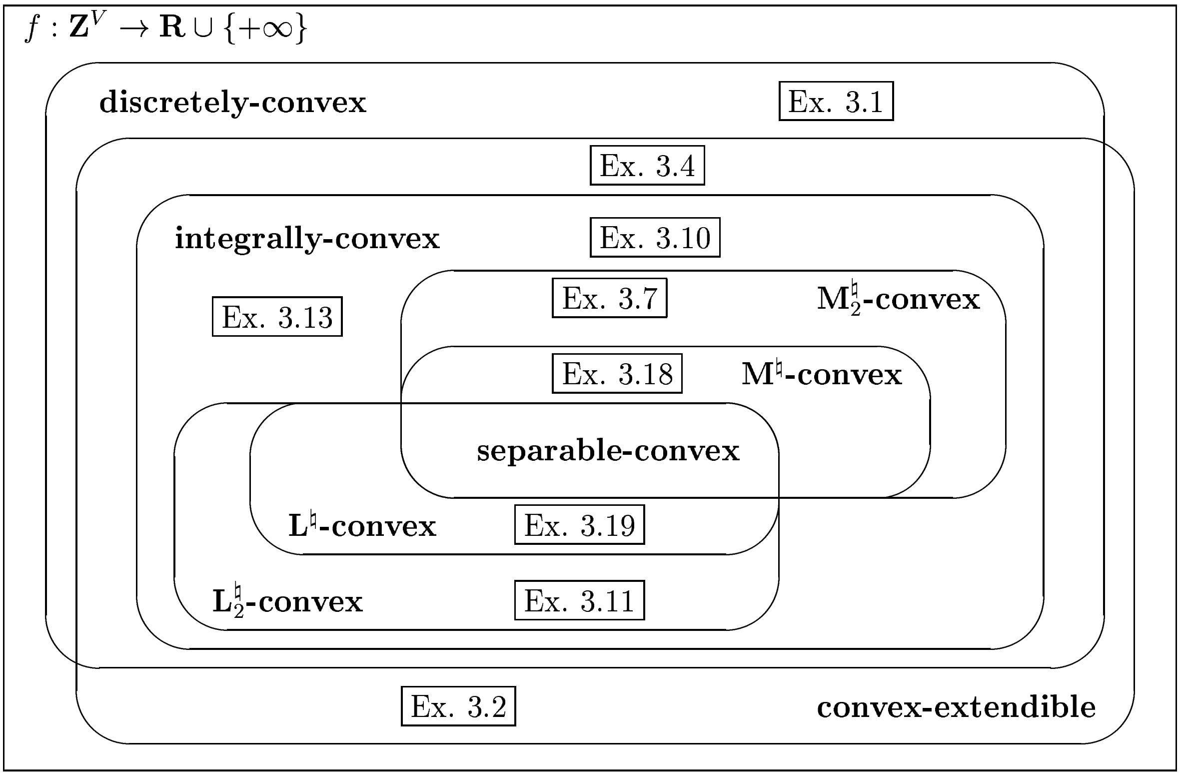

Figure 1: Relationship among discrete convex functions
($\mathrm{M}^\natural$-convex $\cap$ $\mathrm{L}^\natural$-convex $=$ $\mathrm{M}_2^\natural$-convex $\cap$ $\mathrm{L}_2^\natural$-convex $=$ separable-convex)

is discretely-convex. It is not convex-extendible since

$$\frac{1}{3}(1,2,0) + \frac{1}{3}(0,1,2) + \frac{1}{3}(2,0,1) = (1,1,1) \notin S. \qquad \square$$

Example 3.2 The set $S = \{(0,0),(2,1)\}$ is an example of a convex-extendible set which is not discretely-convex. $\square$

Favati–Tardella [1] showed that an integrally-convex function is discretely-convex. It is obvious from its definition that an integrally-convex function is also convex-extendible.

Theorem 3.3 *An integrally-convex function is both discretely-convex and convex-extendible.*

The converse of Theorem 3.3 does not hold in general.

Example 3.4 The set $S = \{(0,0),(1,0),(2,1)\}$ is both discretely-convex and convex-extendible, but not integrally-convex. $\square$

Remark 3.5 Recall that any function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \subseteq \{0,1\}^V$ can be extended to a convex function. Therefore, such a function f is integrally-convex, which implies that f is also discretely-convex. Hence, there is no meaning to introduce these concepts for set functions $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$. $\square$

We review some properties of M-convex/L-convex functions. Let $\mathcal{M}$ (resp. $\mathcal{M}^\natural$, $\mathcal{M}_2$, $\mathcal{M}_2^\natural$) denote the class of M-convex (resp. M$^\natural$-convex, M$_2$-convex, M$_2^\natural$-convex) functions with nonempty effective domain. Similarly, let $\mathcal{L}$ (resp. $\mathcal{L}^\natural$, $\mathcal{L}_2$, $\mathcal{L}_2^\natural$) be the class of L-convex (resp. L$^\natural$-convex, L$_2$-convex, L$_2^\natural$-convex) functions with nonempty effective domain. For any class of functions $\mathcal{F}$, we denote by $(\mathcal{F})_n$ $(n \geq 1)$ the subclass of $\mathcal{F}$ consisting of functions defined over the n-dimensional integer lattice, and by $\mathcal{F}[\mathbf{Z}]$ the subclass of $\mathcal{F}$ consisting of integer-valued functions. The correspondence by projection or restriction (cf. (2.6), (2.7)) is indicated by "$\simeq$". For a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$, the *conjugate function* $f^\bullet : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ of f is defined by

$$f^\bullet(p) = \sup_{x \in \mathbf{Z}^V} \{\langle p, x\rangle - f(x)\} \qquad (p \in \mathbf{Z}^V). \tag{3.1}$$

This operation is called the *(discrete) Fenchel-Legendre transformation.* Note that $f^\bullet = -\infty$ if $\mathrm{dom}\, f = \emptyset$.

Theorem 3.6
(i) $\mathcal{M} \subseteq \mathcal{M}^\natural \simeq \mathcal{M}$ *and* $\mathcal{M}_2 \subseteq \mathcal{M}_2^\natural \simeq \mathcal{M}_2$*; to be more specific,*

$$(\mathcal{M})_n \subseteq (\mathcal{M}^\natural)_n \simeq (\mathcal{M})_{n+1} \subseteq (\mathcal{M}^\natural)_{n+1}, \qquad (\mathcal{M}_2)_n \subseteq (\mathcal{M}_2^\natural)_n \simeq (\mathcal{M}_2)_{n+1} \subseteq (\mathcal{M}_2^\natural)_{n+1}.$$

(ii) $\mathcal{L} \subseteq \mathcal{L}^\natural \simeq \mathcal{L}$ *and* $\mathcal{L}_2 \subseteq \mathcal{L}_2^\natural \simeq \mathcal{L}_2$*; to be more specific,*

$$(\mathcal{L})_n \subseteq (\mathcal{L}^\natural)_n \simeq (\mathcal{L})_{n+1} \subseteq (\mathcal{L}^\natural)_{n+1}, \qquad (\mathcal{L}_2)_n \subseteq (\mathcal{L}_2^\natural)_n \simeq (\mathcal{L}_2)_{n+1} \subseteq (\mathcal{L}_2^\natural)_{n+1}.$$

(iii) *The following pairs of function classes are conjugate to each other under the Fenchel-Legendre transformation:*

$$\begin{array}{ll} (\mathcal{M}[\mathbf{Z}])_n \longleftrightarrow (\mathcal{L}[\mathbf{Z}])_n, & (\mathcal{M}_2[\mathbf{Z}])_n \longleftrightarrow (\mathcal{L}_2[\mathbf{Z}])_n, \\ (\mathcal{M}^\natural[\mathbf{Z}])_n \longleftrightarrow (\mathcal{L}^\natural[\mathbf{Z}])_n, & (\mathcal{M}_2^\natural[\mathbf{Z}])_n \longleftrightarrow (\mathcal{L}_2^\natural[\mathbf{Z}])_n. \end{array}$$

In the following, we mainly consider M$^\natural$/M$_2^\natural$/L$^\natural$/L$_2^\natural$-convex functions instead of M/M$_2$/L/L$_2$-convex functions.

The class of M$_2^\natural$-convex functions properly contains that of M$^\natural$-convex functions, which follows from the definition and the following example.

Example 3.7 This is an example of an M$_2^\natural$-convex set which is not M$^\natural$-convex. The set

$$S = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,0,1)\}$$

is an M$_2^\natural$-convex set represented as the intersection of M$^\natural$-convex sets $S_1 = S \cup \{(0,1,1)\}$ and $S_2 = S \cup \{(1,1,0)\}$. S is not an M$^\natural$-convex set since the property (M$^\natural$-EXC) does not hold for $x = (1,0,1)$, $y = (0,1,0)$, and $u =$"1". Note that S_1 and S_2 each correspond to the family of independent sets of a matroid. □

The class of $\mathrm{M}_2^\natural$-convex functions is properly contained in the intersection of the classes of integrally-convex/supermodular functions, as shown in the following theorems and example.

Theorem 3.8 *An $M_2^\natural$-convex function is supermodular. In particular, an $M^\natural$-convex function is supermodular.*

Theorem 3.9 *An $M_2^\natural$-convex function is integrally-convex. In particular, an $M^\natural$-convex function is integrally-convex.*

The proofs of Theorems 3.8 and 3.9 are given in Sections 5.1 and 5.2, respectively.

Example 3.10 This is an example of a set which is both integrally-convex and supermodular, and not $\mathrm{M}_2^\natural$-convex. The set $S = \{(1,0,0,0),(0,1,1,1)\}$ is obviously integrally-convex and supermodular. Suppose that S is an $\mathrm{M}_2^\natural$-convex set expressed as $S = S_1 \cap S_2$ for some $\mathrm{M}^\natural$-convex sets $S_1, S_2 \subseteq \mathbf{Z}^4$. Since $(1,0,0,0),(0,1,1,1) \in S_1 \cap S_2$, the property ($\mathrm{M}^\natural$-EXC) implies the existence of $x \in S_1 \cap S_2$ with $\sum_{i=1}^4 x_i = 2$, a contradiction. □

We next consider the classes of $\mathrm{L}^\natural$-convex/$\mathrm{L}_2^\natural$-convex functions. It is clear from the definition and the following example that the class of $\mathrm{L}_2^\natural$-convex functions properly contains that of $\mathrm{L}^\natural$-convex functions.

Example 3.11 This is an example of an $\mathrm{L}_2^\natural$-convex set which is not $\mathrm{L}^\natural$-convex. The set

$$\{(0,0,0),(0,1,1),(1,1,0),(1,2,1)\} \ (= \{(0,0,0),(1,1,0)\} + \{(0,0,0),(0,1,1)\})$$

is an $\mathrm{L}_2^\natural$-convex set. It is not $\mathrm{L}^\natural$-convex, since the mid-point convexity (2.8) does not hold for $(1,1,0)$ and $(0,1,1)$. □

The class of $\mathrm{L}_2^\natural$-convex functions is properly contained in that of integrally-convex functions, which is shown by the following theorem and example.

Theorem 3.12 *An $L_2^\natural$-convex function is integrally-convex. In particular, an $L^\natural$-convex function is integrally-convex.*

Proof. The proof is given in Section 5.3. □

Example 3.13 This is an example of an integrally-convex set which is neither $\mathrm{M}_2^\natural$-convex nor $\mathrm{L}_2^\natural$-convex. The set $S = \{(0,0,0),(1,0,0),(0,1,0),(1,0,1)\}$ is an integrally-convex set. It is not an $\mathrm{L}_2^\natural$-convex set by Lemma 3.14 below since it contains two maximal vectors. S is not supermodular, and therefore not $\mathrm{M}_2^\natural$-convex by Theorem 3.8. □

Lemma 3.14 *A bounded $L_2^\natural$-convex set has the unique minimal and maximal vectors.*

Proof. First note that a bounded L$^\natural$-convex set contains the unique minimal (resp. maximal) vector. A bounded L$_2^\natural$-convex set is represented as the Minkowski-sum of two bounded L$^\natural$-convex sets, and the unique minimal (resp. maximal) vector is the sum of the unique minimal (resp. maximal) vectors of the summands. □

From Theorem 3.12 and its definition we see that any L$^\natural$-convex function is integrally-convex and submodular. In fact, the converse of this statement holds true.

Theorem 3.15 ([6]) *A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is $L^\natural$-convex if and only if it is integrally-convex and submodular. In particular, a function f with $\mathrm{dom}\, f \subseteq \{0,1\}^V$, is $L^\natural$-convex if and only if it is submodular.*

Finally, we characterize separable-convex functions as those which are at the same time M$_2^\natural$-convex and L$_2^\natural$-convex.

Lemma 3.16 ([6, 13]) *A separable-convex function is both $M^\natural$-convex and $L^\natural$-convex.*

Therefore, any separable-convex function is both M$_2^\natural$-convex and L$_2^\natural$-convex. In fact, the converse of this statement holds true.

Theorem 3.17 *A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is separable-convex if and only if it is both $M_2^\natural$-convex and $L_2^\natural$-convex.*

Proof. The proof is given in Section 5.4. □

The class of separable-convex functions is properly contained in the classes of M$^\natural$-convex/L$^\natural$-convex functions, respectively.

Example 3.18 The set $S = \{(1,0),(0,1)\}$ is M$^\natural$-convex and not separable-convex. □

Example 3.19 The set $S = \{(0,0),(1,1)\}$ is L$^\natural$-convex and not separable-convex. □

Remark 3.20 None of submodularity, supermodularity, and modularity imply discrete convexity. For example, the set $\{(0,0),(2,0),(0,2),(2,2)\}$, which is modular (hence both submodular and supermodular), is neither discretely-convex nor convex-extendible. □

4 Operations for Discrete Convex Functions

In this section, we discuss some operations for discrete convex functions and the corresponding operations for discrete convex sets. We show proofs and examples to clarify whether each discrete convexity is closed under such operations. We also investigate level sets of discrete convex functions. The results in this section is summarized in Table 1.

Table 1: Operations for discrete convex sets and functions

	disc.-conv.	conv.-ext.	int.-conv.	sep.-conv.
$f_1 + f_2$	$\times$ (Ex.4.2,4.4)	$\bigcirc$ (Th. 4.1)	$\times$ (Ex. 4.4)	$\bigcirc$
$S_1 \cap S_2$	$\times$ (Ex. 4.4)	$\bigcirc$ (Th. 4.1)	$\times$ (Ex. 4.4)	$\bigcirc$
f+sep.-conv.	$\times$ (Ex. 4.2)	$\bigcirc$ (Th. 4.1)	$\bigcirc$ (Th. 4.5)	$\bigcirc$
$S \cap [a, b]$	$\bigcirc$ (Th. 4.3)	$\bigcirc$ (Th. 4.1)	$\bigcirc$ (Th. 4.5)	$\bigcirc$
f+ affine	$\times$ (Ex. 4.2)	$\bigcirc$ (Th. 4.1)	$\bigcirc$ (Th. 4.5)	$\bigcirc$
$f_1 \Box f_2$	$\times$ (Ex. 4.12)	$\times$ (Ex. 4.12)	$\times$ (Ex. 4.12)	$\bigcirc$
$S_1 + S_2$	$\times$ (Ex. 4.12)	$\times$ (Ex. 4.12)	$\times$ (Ex. 4.12)	$\bigcirc$
$f^\bullet$	$\times$ (Ex. 4.14)	$\bigcirc$ (Ex. 4.13)	$\times$ (Ex. 4.15)	$\bigcirc$
$\mathrm{L}(f, \lambda)$	$\bigcirc$ (Th. 4.16)	$\bigcirc$ (Th. 4.17)	$\times$ (Ex. 4.18)	$\times$ (Ex. 4.18)
dom f	$\bigcirc$ (Th. 4.16)	$\bigcirc$ (Th. 4.17)	$\bigcirc$ (Th. 4.21)	$\bigcirc$
arg min f	$\bigcirc$ (Th. 4.16)	$\bigcirc$ (Th. 4.17)	$\bigcirc$ (Th. 4.21)	$\bigcirc$

	$\mathrm{M}^\natural_2$-conv.	$\mathrm{L}^\natural_2$-conv.	$\mathrm{M}^\natural$-conv.	$\mathrm{L}^\natural$-conv.
$f_1 + f_2$	$\times$ (Ex. 4.6)	$\times$ (Ex.4.4,4.8)	$\times$ ($\mathrm{M}^\natural_2$-conv.)	$\bigcirc$
$S_1 \cap S_2$	$\times$ (Ex. 4.6)	$\times$ (Ex.4.4,4.8)	$\times$ ($\mathrm{M}^\natural_2$-conv.)	$\bigcirc$
f+sep.-conv.	$\bigcirc$ (Th. 4.7)	$\times$ (Ex. 4.8)	$\bigcirc$ (Th. 4.7)	$\bigcirc$
$S \cap [a, b]$	$\bigcirc$ (Th. 4.7)	$\times$ (Ex. 4.8)	$\bigcirc$ (Th. 4.7)	$\bigcirc$
f+ affine	$\bigcirc$ (Th. 4.7)	$\bigcirc$ (Th. 4.9)	$\bigcirc$ (Th. 4.7)	$\bigcirc$
$f_1 \Box f_2$	$\times$ (Ex. 4.11)	$\times$ (Ex. 4.12)	$\bigcirc$ (Th. 4.10)	$\times$ ($\mathrm{L}^\natural_2$-conv.)
$S_1 + S_2$	$\times$ (Ex. 4.11)	$\times$ (Ex. 4.12)	$\bigcirc$ (Th. 4.10)	$\times$ ($\mathrm{L}^\natural_2$-conv.)
$f^\bullet$	$\times$ ($\mathrm{L}^\natural_2$-conv.)	$\times$ ($\mathrm{M}^\natural_2$-conv.)	$\times$ ($\mathrm{L}^\natural$-conv.)	$\times$ ($\mathrm{M}^\natural$-conv.)
$\mathrm{L}(f, \lambda)$	$\times$ (Ex.4.18,4.19)	$\times$ (Ex.4.18,4.20)	$\times$ (Ex.4.18,4.19)	$\times$ (Ex.4.18,4.20)
dom f	$\bigcirc$ (Th. 4.25)	$\bigcirc$ (Th. 4.26)	$\bigcirc$ (Th. 4.22)	$\bigcirc$ (Th. 4.23)
arg min f	$\bigcirc$ (Th. 4.25)	$\bigcirc$ (Th. 4.26)	$\bigcirc$ (Th. 4.22)	$\bigcirc$ (Th. 4.23)

4.1 Addition of Two Functions

It may be clear from the definitions that the classes of L$^\natural$-convex/separable-convex functions are closed under addition. Also, the class of convex-extendible functions is closed under addition.

Theorem 4.1 *The sum of two convex-extendible functions is convex-extendible.*

Proof. Let f_1, f_2 be convex-extendible functions. Since $\overline{f_1} + \overline{f_2}$ is a convex extension of $f_1 + f_2$, the function $f_1 + f_2$ is convex-extendible by Lemma 2.3. □

The class of discretely-convex functions is not closed under addition.

Example 4.2 The sum of a discretely-convex function and an affine function is not necessarily a discretely-convex function. Let $f_i : \mathbf{Z}^2 \to \mathbf{Z} \cup \{+\infty\}$ $(i = 1, 2)$ be functions defined by

$$\begin{aligned} f_1(x_1, x_2) &= \begin{cases} 0 & \text{if } (x_1, x_2) \in \{(0,0), (1,0), (2,1)\}, \\ 2 & \text{if } (x_1, x_2) = (1,1), \\ +\infty & \text{otherwise}, \end{cases} \\ f_2(x_1, x_2) &= x_1 - 2x_2 \qquad ((x_1, x_2) \in \mathbf{Z}^2), \end{aligned}$$

where f_1 is a discretely-convex function. The function $f = f_1 + f_2$ is given by

$$f(x_1, x_2) = \begin{cases} 0 & \text{if } (x_1, x_2) \in \{(0,0), (2,1)\}, \\ 1 & \text{if } (x_1, x_2) \in \{(1,0), (1,1)\}, \\ +\infty & \text{otherwise}, \end{cases}$$

which is not discretely-convex since $\min\{f(1,0), f(1,1)\} = 1 > 0 = \{f(0,0) + f(2,1)\}/2$. □

Although the class of discretely-convex functions is not closed under the addition of a separable-convex function, the class of discretely-convex sets is obviously closed under the corresponding operation for sets, i.e., the intersection with a separable-convex set (an interval).

Theorem 4.3 *The intersection of a discretely-convex set and a separable-convex set is discretely-convex.*

The classes of integrally-convex/L$^\natural_2$-convex functions are not closed under addition, as the following example shows.

Example 4.4 The intersection of two L$^\natural_2$-convex sets is not necessarily discretely-convex. The sets

$$\begin{aligned} D_1 &= \{(0,0,0), (0,1,1), (1,1,0), (1,2,1)\} \\ &= \{(0,0,0), (0,1,1)\} + \{(0,0,0), (1,1,0)\}, \\ D_2 &= \{(0,0,0), (0,1,0), (1,1,1), (1,2,1)\} \\ &= \{(0,0,0), (0,1,0)\} + \{(0,0,0), (1,1,1)\} \end{aligned}$$

are both L$^\natural_2$-convex. We have $D_1 \cap D_2 = \{(0,0,0), (1,2,1)\}$, which is not discretely-convex since $(D_1 \cap D_2) \cap \mathrm{N}_0(x) = \emptyset$ for $x = (1/2, 1, 1/2)$. □

Though the class of integrally-convex functions is not closed under addition, it is closed under the addition of a separable-convex function.

Theorem 4.5 *The sum of an integrally-convex function and a separable-convex function is integrally-convex.*

Proof. Define $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by $f(x) = f_0(x) + \sum_{v \in V} f_v(x(v))$ $(x \in \mathbf{Z}^V)$, where $f_0 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is an integrally-convex function and $f_v : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$ is a one-dimensional convex function for each $v \in V$. As shown below, we have

$$\tilde{f}(x) = \tilde{f}_0(x) + \sum_{v \in V} \tilde{f}_v(x(v)) \qquad (x \in \mathbf{R}^V). \tag{4.1}$$

Since the RHS of (4.1) is equal to $\overline{f}_0(x) + \sum_{v \in V} \overline{f}_v(x(v))$, the function $\tilde{f}$ is convex. Thus, f is integrally-convex by its definition.

We now prove (4.1). Let $x \in \mathbf{R}^V$, and $\lambda = (\lambda_y \mid y \in \mathrm{N}_0(x))$ be any vector such that

$$\sum_{y \in \mathrm{N}_0(x)} \lambda_y y = x, \qquad \sum_{y \in \mathrm{N}_0(x)} \lambda_y = 1, \quad \lambda_y \geq 0 \ (\forall y \in \mathrm{N}_0(x)). \tag{4.2}$$

We have

$$\begin{aligned}
\sum_{y \in \mathrm{N}_0(x)} \lambda_y f(y) &= \sum_{y \in \mathrm{N}_0(x)} \lambda_y f_0(y) + \sum_{y \in \mathrm{N}_0(x)} \lambda_y \sum_{v \in V} f_v(y(v)) \\
&= \sum_{y \in \mathrm{N}_0(x)} \lambda_y f_0(y) + \sum_{v \in V} \sum_{y \in \mathrm{N}_0(x)} \lambda_y f_v(y(v)) \ = \ \sum_{y \in \mathrm{N}_0(x)} \lambda_y f_0(y) + \sum_{v \in V} \tilde{f}_v(x(v)),
\end{aligned}$$

where the last equality is due to the fact that a function $\tilde{f}_v$ is linear in each interval $[\alpha, \alpha + 1]$ $(\alpha \in \mathbf{Z})$. Therefore, it holds that

$$\begin{aligned}
\tilde{f}(x) &= \inf_{\lambda} \left\{ \sum_{y \in \mathrm{N}_0(x)} \lambda_y f(y) \,\middle|\, \lambda = (\lambda_y \mid y \in \mathrm{N}_0(x)) \text{ satisfies } (4.2) \right\} \\
&= \inf_{\lambda} \left\{ \sum_{y \in \mathrm{N}_0(x)} \lambda_y f_0(y) \,\middle|\, \lambda = (\lambda_y \mid y \in \mathrm{N}_0(x)) \text{ satisfies } (4.2) \right\} + \sum_{v \in V} \tilde{f}_v(x(v)) \\
&= \tilde{f}_0 + \sum_{v \in V} \tilde{f}_v(x(v)).
\end{aligned}$$

□

The sum of two M$^\natural$-convex functions is not M$^\natural$-convex in general (see Example 3.7) but M$_2^\natural$-convex by definition. The class of M$_2^\natural$-convex functions is not closed under addition as shown in Example 4.6 below, but closed under the addition of a separable-convex function.

Example 4.6 The intersection of three M$^\natural$-convex sets is not necessarily M$_2^\natural$-convex. For any $i, j \in \{1, 2, 3, 4, 5\}$, we denote $\chi_{ij} = \chi_{\{i,j\}} \in \mathbf{Z}^5$. Let

$$\begin{aligned}
&S = \{x_{12}, x_{13}, x_{34}, x_{35}, x_{45}\}, \\
&S_1 = S \cup \{x_{14}, x_{23}, x_{25}\}, \quad S_2 = S \cup \{x_{15}, x_{23}, x_{24}\}, \quad S_3 = S \cup \{x_{14}, x_{24}, x_{25}\}.
\end{aligned}$$

Each S_i is M-convex, and therefore M$^\natural$-convex. In fact, each S_i corresponds to the basis family of a certain graphic matroid. As shown below, $S = S_1 \cap S_2 \cap S_3$ is not M$_2$-convex, which implies that S is not M$_2^\natural$-convex. Note that a set $S \subseteq \mathbf{Z}^V$ is M$_2$-convex if and only if S is M$_2^\natural$-convex and $x(V) = y(V)$ for any $x, y \in S$.

Suppose $S = S'_1 \cap S'_2$ for two M-convex sets $S'_1, S'_2 \subseteq \mathbf{Z}^V$. Then, the property (M-EXC) implies that:

$$\text{each } S'_i \text{ must contain} \begin{cases} \text{either } \{x_{25}, x_{14}\} \text{ or } \{x_{24}, x_{15}\} & \text{(by (M-EXC) for } x_{45}, x_{12}), \\ \text{either } \{x_{13}, x_{24}\} \text{ or } \{x_{14}, x_{23}\} & \text{(by (M-EXC) for } x_{12}, x_{34}), \\ \text{either } \{x_{13}, x_{25}\} \text{ or } \{x_{23}, x_{15}\} & \text{(by (M-EXC) for } x_{35}, x_{12}), \\ \text{either } \{x_{35}, x_{14}\} \text{ or } \{x_{34}, x_{15}\} & \text{(by (M-EXC) for } x_{45}, x_{13}). \end{cases}$$

Hence, S'_1 and S'_2 must contain a common vector which is not in S, a contradiction. □

Theorem 4.7
(i) *The sum of an M$^\natural$-convex function and a separable-convex function is M$^\natural$-convex.*
(ii) *The sum of an M$_2^\natural$-convex function and a separable-convex function is M$_2^\natural$-convex.*

Proof. (i) is shown in [12, Example 4.2], and (ii) is immediate from (i). □

The class of L$_2^\natural$-convex functions is not closed under addition.

Example 4.8 This example shows that the intersection of an L$_2^\natural$-convex set and a separable-convex set (an interval) is not L$_2^\natural$-convex. The set

$$S = \{(0,0,0), (1,1,0), (0,1,1), (1,2,1)\} \ (= \{(0,0,0), (1,1,0)\} + \{(0,0,0), (0,1,1)\})$$

is L$_2^\natural$-convex. We have $S \cap \{0,1\}^3 = \{(0,0,0), (1,1,0), (0,1,1)\}$, which is not L$_2^\natural$-convex by Lemma 3.14. □

As shown in the example above, the class of L$_2^\natural$-convex functions is not closed under the addition of a separable-convex function. It is closed, however, under the addition of an affine function.

Theorem 4.9 *The sum of an L$_2^\natural$-convex function and an affine function is L$_2^\natural$-convex.*

Proof. Let $f_i : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ $(i = 1, 2)$ be L$^\natural$-convex functions, $p \in \mathbf{R}^V$, and $\alpha \in \mathbf{R}$. Put $f(x) = (f_1 \Box f_2)(x) + (\langle p, x \rangle + \alpha)$ $(x \in \mathbf{Z}^V)$. Then, we have

$$\begin{aligned} f(x) &= \inf_{x_1, x_2 \in \mathbf{Z}^V} \{f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x\} + (\langle p, x \rangle + \alpha) \\ &= \inf_{x_1, x_2 \in \mathbf{Z}^V} \{(f_1(x_1) + \langle p, x_1 \rangle + \alpha) + (f_2(x_2) + \langle p, x_2 \rangle) \mid x_1 + x_2 = x\}. \end{aligned}$$

Since the functions $f_1 + \langle p, \cdot \rangle + \alpha$ and $f_2 + \langle p, \cdot \rangle$ are L$^\natural$-convex, the function f is L$_2^\natural$-convex. □

4.2 Convolution

The convolution of two M$^\natural$-convex functions is known to be M$^\natural$-convex.

Theorem 4.10 ([10, Theorem 6.10], [11, Theorem 5.8]) *For two M$^\natural$-convex functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, the convolution $f_1 \Box f_2$ is also M$^\natural$-convex provided that $f_1 \Box f_2 > -\infty$.*

Obviously, the class of separable-convex functions is closed under convolution.

The following two examples show that the classes of discretely-convex/convex-extendible/integrally-convex/M$_2^\natural$-convex/L$_2^\natural$-convex functions are not closed under convolution. Recall that the convolution of two indicator functions of sets corresponds to the Minkowski-sum of the two sets.

Example 4.11 The Minkowski-sum of an M$_2^\natural$-convex set and an M$^\natural$-convex set is not necessarily M$_2^\natural$-convex. The set $S_1 = \{(0,0,1,1), (1,1,0,0), (1,0,1,0)\}$ is an M$_2$-convex set expressed as the intersection of two M-convex sets $S_1 \cup \{(0,1,0,1)\}$ and $S_1 \cup \{(0,1,1,0), (1,0,0,1)\}$, and $S_2 = \{(1,0,0,1), (0,1,0,1)\}$ is an M-convex set. The Minkowski-sum $S = S_1 + S_2$ is given by

$$S = \{(0,1,1,2), (1,1,1,1), (2,1,0,1), (1,0,1,2), (2,0,1,1), (1,2,0,1)\}.$$

Suppose $S = S_1' \cap S_2'$ for two M-convex sets $S_1', S_2' \subseteq \mathbf{Z}^V$. Since $x = (1,0,1,2)$ and $y = (1,2,0,1)$ are contained in $S_1' \cap S_2'$, (M-EXC) implies that $(1,1,0,2) \in S_1' \cap S_2' = S$, a contradiction. Hence, S is not M$_2$-convex, and therefore it is not M$_2^\natural$-convex since $\sum_{i=1}^{4} x_i = 4$ for $x \in S$. □

Example 4.12 The Minkowski-sum of three L$^\natural$-convex sets is neither discretely-convex nor convex-extendible in general. Each of

$$S_1 = \{(0,0,0), (1,1,0)\}, \quad S_2 = \{(0,0,0), (0,1,1)\}, \quad S_3 = \{(0,0,0), (1,0,1)\}$$

is L$^\natural$-convex. The Minkowski-sum $S = S_1 + S_2 + S_3$ given by

$$S = \{(0,0,0), (0,1,1), (1,1,0), (1,0,1), (2,1,1), (1,1,2), (1,2,1), (2,2,2)\}$$

is not convex-extendible since $(1,1,1) \in \overline{S} - S$, nor discretely-convex since $(x_1 + x_2)/2 = (1,1,1) \notin S$ for $x_1 = (1,1,0) \in S$ and $x_2 = (1,1,2) \in S$. □

The convolution of two L$^\natural$-convex functions, which is called L$_2^\natural$-convex by definition, is not necessarily L$^\natural$-convex, as shown in Example 3.11.

4.3 Fenchel–Legendre Transformation

In this section, we consider only functions $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$, since if $\mathrm{dom}\, f = \emptyset$ then $f^\bullet = -\infty$.

It is clear that the class of separable-convex functions is closed under the Fenchel-Legendre transformation. This is also the case with the class of convex-extendible functions. Moreover, the conjugate of any function is convex-extendible.

Theorem 4.13 *For any function* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom}\, f \neq \emptyset$*, its conjugate* $f^\bullet : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *is a convex-extendible function.*

Proof. Define a function $f^\bullet_{\mathbf{R}} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ by (3.1), where $p \in \mathbf{R}^V$. Then, it is easy to see that the function $f^\bullet_{\mathbf{R}}$ is a convex extension of $f^\bullet$. □

As shown in Theorem 3.6 (iii), the classes of integer-valued M$^\natural$-convex/L$^\natural$-convex functions are conjugate to each other, and the classes of integer-valued M$^\natural_2$-convex/L$^\natural_2$-convex functions are conjugate to each other. Therefore, the classes of (real-valued) M$^\natural$-convex/L$^\natural$-convex/M$^\natural_2$-convex/L$^\natural_2$-convex functions are not closed under the Fenchel-Legendre transformation.

The classes of discretely-convex/integrally-convex functions are not closed under the Fenchel-Legendre transformation.

Example 4.14 The conjugate of a discretely-convex function is not necessarily discretely-convex. The set $S = \{(0,0), (-1,0), (1,1)\}$ is a discretely-convex set and therefore its indicator function $\delta_S : \mathbf{Z}^2 \to \{0, +\infty\}$ is discretely-convex. The conjugate of δ_S is given by

$$\delta^\bullet_S(p_1, p_2) = \max\{0, -p_1, p_1 + p_2\} \qquad ((p_1, p_2) \in \mathbf{Z}^2),$$

which is not discretely-convex since

$$\min\{\delta^\bullet_S(-1,1), \delta^\bullet_S(0,1)\} = 1 > \frac{1}{2} = \frac{1}{2}\delta^\bullet_S(-1,2) + \frac{1}{2}\delta^\bullet_S(0,0).$$

□

Example 4.15 The conjugate of an integrally-convex function is not necessarily integrally-convex. The set $S = \{(1,1,0,0), (0,1,1,0), (1,0,1,0), (0,0,0,1)\}$ is an integrally-convex set and therefore its indicator function $\delta_S : \mathbf{Z}^4 \to \{0, +\infty\}$ is integrally-convex. The conjugate of δ_S is given by

$$\delta^\bullet_S(p_1, p_2, p_3, p_4) = \max\{p_1 + p_2, p_2 + p_3, p_1 + p_3, p_4\} \qquad (p \in \mathbf{Z}^4).$$

The convex closure $\overline{g}$ of $g = \delta^\bullet_S$ is given by the same expression for $p \in \mathbf{R}^4$. Since $\overline{g}(1/2, 1/2, 1/2, 1) = 1 < 3/2 = \tilde{g}(1/2, 1/2, 1/2, 1)$, g is not integrally-convex. □

4.4 Level Sets

For a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and a value $\lambda \in \mathbf{R} \cup \{+\infty\}$, the *level set* $\mathrm{L}(f, \lambda)$ is defined by $\mathrm{L}(f, \lambda) = \{x \in \mathbf{Z}^V \mid f(x) \leq \lambda\}$. The effective domain $\mathrm{dom}\, f$ and the set of minimizers $\arg\min f$ can be seen as special cases of level sets with $\lambda = +\infty$ and $\lambda = \min f$, respectively.

Level sets of a discretely-convex (resp. convex-extendible) function are discretely-convex (resp. convex-extendible) sets.

Theorem 4.16 *For a discretely-convex function* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *and a value* $\lambda \in \mathbf{R} \cup \{+\infty\}$*, the level set* $\mathrm{L}(f, \lambda)$ *is a discretely-convex set.*

Proof. Let $x', x'' \in \mathrm{L}(f, \lambda)$ and $\alpha \in [0, 1]$. Then, we have

$$\min\{f(y) \mid y \in \mathrm{N}_0(\alpha x' + (1-\alpha)x'')\} \leq \alpha f(x') + (1-\alpha)f(x'') \leq \lambda.$$

This implies that $\mathrm{N}_0(\alpha x' + (1-\alpha)x'') \cap \mathrm{L}(f, \lambda) \neq \emptyset$. Hence, $\mathrm{L}(f, \lambda)$ is a discretely-convex set. □

Theorem 4.17 *For a convex-extendible function* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *and a value* $\lambda \in \mathbf{R} \cup \{+\infty\}$*, the level set* $\mathrm{L}(f, \lambda)$ *is a convex-extendible set.*

Proof. Since $\overline{f}$ is a convex function, the level set $\{x \in \mathbf{R}^V \mid \overline{f}(x) \leq \lambda\}$ is a convex set. Since $\mathrm{L}(f, \lambda) = \{x \in \mathbf{R}^V \mid \overline{f}(x) \leq \lambda\} \cap \mathbf{Z}^V$, the level set $\mathrm{L}(f, \lambda)$ is a convex-extendible set. □

For an integrally-convex/separable-convex/M$^\natural$-convex/M$_2^\natural$-convex/L$^\natural$-convex/L$_2^\natural$-convex function, a level set does not necessarily have the corresponding discrete convexity.

Example 4.18 A level set of a linear function is not necessarily integrally-convex. For a linear function $f : \mathbf{Z}^2 \to \mathbf{Z}$ defined by $f(x_1, x_2) = x_1 + 2x_2$ $((x_1, x_2) \in \mathbf{Z}^2)$, we have $\mathrm{L}(f, 0) = \{(x_1, x_2) \in \mathbf{Z}^2 \mid x_1 + 2x_2 \leq 0\}$, which is not an integrally-convex set. □

Example 4.19 A level set of a linear function defined over $\{0,1\}^V$ is neither M$^\natural$-convex nor M$_2^\natural$-convex in general. For a linear function $f : \mathbf{Z}^2 \to \mathbf{Z}$ defined by

$$f(x_1, x_2) = \begin{cases} -x_1 + x_2 & ((x_1, x_2) \in \{0,1\}^2), \\ +\infty & (\text{otherwise}), \end{cases}$$

we have $\mathrm{L}(f, 0) = \{(0,0), (1,0), (1,1)\}$, which is not M$_2^\natural$-convex since it is not a supermodular set. □

Example 4.20 A level set of a linear function defined over $\{0,1\}^V$ is neither L$^\natural$-convex nor L$_2^\natural$-convex in general. For a linear function $f : \mathbf{Z}^2 \to \mathbf{Z}$ defined by

$$f(x_1, x_2) = \begin{cases} x_1 + x_2 & ((x_1, x_2) \in \{0,1\}^2), \\ +\infty & (\text{otherwise}), \end{cases}$$

we have $\mathrm{L}(f, 1) = \{(0,0), (1,0), (0,1)\}$, which is not L$_2^\natural$-convex since it is not a submodular set. □

The effective domain and the set of minimizers have the corresponding discrete convexity for an integrally-convex/separable-convex/M$^\natural$-convex/M$_2^\natural$-convex/L$^\natural$-convex/L$_2^\natural$-convex function.

Theorem 4.21 *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an integrally-convex function.*
(i) $\mathrm{dom}\, f$ *is an integrally-convex set.* (ii) $\arg\min f$ *is an integrally-convex set.*

Proof. (i): It suffices to show (2.5) for $S = \mathrm{dom}\, f$. Since $\overline{f} = \widetilde{f}$, we have

$$\overline{\mathrm{dom}\, f} \cap \overline{\mathrm{N}_0(x)} = \mathrm{dom}\, \overline{f} \cap \overline{\mathrm{N}_0(x)} = \mathrm{dom}\, \widetilde{f} \cap \overline{\mathrm{N}_0(x)} = \overline{\mathrm{dom}\, f \cap \mathrm{N}_0(x)},$$

where the last equality is by the definition of $\widetilde{f}$.

(ii): It suffices to show (2.5) for $S = \arg\min f$. The inclusion $\overline{S \cap \mathrm{N}_0(x)} \subseteq \overline{S} \cap \overline{\mathrm{N}_0(x)}$ holds obviously. For $x' \in \overline{S} \cap \overline{\mathrm{N}_0(x)}$, we have $\inf f = \overline{f}(x') = \widetilde{f}(x')$. Therefore, $x' \in \overline{S \cap \mathrm{N}_0(x)}$. □

Theorem 4.22 ([11, Theorem 4.3, 4.10]) *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an M$^\natural$-convex function.*
(i) $\mathrm{dom}\, f$ *is an M$^\natural$-convex set.* (ii) $\arg\min f$ *is an M$^\natural$-convex set.*

Theorem 4.23 ([11, Theorem 4.16, 4.17]) *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an L$^\natural$-convex function.*
(i) $\mathrm{dom}\, f$ *is an L$^\natural$-convex set.* (ii) $\arg\min f$ *is an L$^\natural$-convex set.*

Theorem 4.24 ([9, Theorem 4.1]) *Let $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be M$^\natural$-convex functions and $x_* \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$. Then,*

$$f_1(x_*) + f_2(x_*) \leq f_1(x) + f_2(x) \qquad (\forall x \in \mathbf{Z}^V)$$

if and only if there exist $p_* \in \mathbf{R}^V$ such that

$$f_1[-p_*](x_*) \leq f_1[-p_*](x) \ (\forall x \in \mathbf{Z}^V), \qquad f_2[p_*](x_*) \leq f_2[p_*](x) \ (\forall x \in \mathbf{Z}^V). \tag{4.3}$$

Theorem 4.25 *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an M$_2^\natural$-convex function.*
(i) $\mathrm{dom}\, f$ *is an M$_2^\natural$-convex set.* (ii) $\arg\min f$ *is an M$_2^\natural$-convex set.*

Proof. We show (ii) only, since (i) is obvious. Let $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be M$^\natural$-convex functions such that $f = f_1 + f_2$, and $x_* \in \arg\min f$. By Theorem 4.24, there exists $p_* \in \mathbf{R}^V$ satisfying (4.3). We have $\arg\min f = \arg\min f_1[-p_*] \cap \arg\min f_2[p_*]$, which is M$_2^\natural$-convex. □

Theorem 4.26 *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an L$_2^\natural$-convex function.*
(i) $\mathrm{dom}\, f$ *is an L$_2^\natural$-convex set.* (ii) $\arg\min f$ *is an L$_2^\natural$-convex set.*

Proof. We show (ii) only, since (i) can be shown similarly. Suppose that f is expressed as $f = f_1 \Box f_2$ for two L$^\natural$-convex functions f_1 and f_2. Since $\inf f = \inf f_1 + \inf f_2$, we have $\arg\min f = \arg\min f_1 + \arg\min f_2$, which implies the L$_2^\natural$-convexity of $\arg\min f$ by Theorem 4.23. □

5 Proofs

5.1 Proof of Theorem 3.8

We show the supermodularity of M$^\natural$-convex functions only. Then, the supermodularity of M$^\natural_2$-convex functions follows immediately since the supermodularity is closed under addition.

Lemma 5.1 *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an M$^\natural$-convex function. Then, we have*

$$f(x + \chi_u) + f(x + \chi_v) \leq f(x) + f(x + \chi_u + \chi_v) \qquad (\forall u, v \in V,\ u \neq v). \tag{5.1}$$

Proof. The claim follows immediately by applying (M$^\natural$-EXC) to $x + \chi_u + \chi_v$, x and u. □

Lemma 5.2 *For any M$^\natural$-convex set $S \subseteq \mathbf{Z}^V$ and any $x, y \in S$ with $x \leq y$, we have $[x, y] \subseteq S$. In particular, an M$^\natural$-convex set is a supermodular set.*

Proof. This follows from the polyhedral description of S (see [4, 5]). □

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an M$^\natural$-convex function. We show the supermodular inequality

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y) \qquad (x, y \in \mathbf{Z}^V) \tag{5.2}$$

by induction on the numbers

$$\begin{aligned} \alpha(x, y) &= \sum\{x(v) - y(v) \mid v \in \mathrm{supp}^+(x - y)\}, \\ \beta(x, y) &= \sum\{y(v) - x(v) \mid v \in \mathrm{supp}^-(x - y)\}. \end{aligned}$$

If $\alpha(x, y) = 0$ or $\beta(x, y) = 0$, then we have either $x \leq y$ or $x \geq y$, and therefore the inequality (5.2) holds obviously. If $\alpha(x, y) = \beta(x, y) = 1$, then (5.2) also holds by Lemma 5.1. Hence, we may assume that $\alpha(x, y) \geq 2$ and $\beta(x, y) \geq 1$. We may also assume that $x \wedge y, x \vee y \in \mathrm{dom}\, f$, which implies $[x \wedge y, x \vee y] \subseteq \mathrm{dom}\, f$ by Theorem 4.22 and Lemma 5.2. Let $u \in \mathrm{supp}^+(x - y)$. Then, the inductive hypothesis implies

$$f(y) - f(x \wedge y) \leq f(y + \chi_u) - f(x \wedge y + \chi_u) \leq f(x \vee y) - f(x).$$

5.2 Proof of Theorem 3.9

From the definition of M$^\natural_2$-convex functions, it suffices to show that any M$_2$-convex function is integrally-convex. First, we consider the special cases of M$_2$-convex sets and M-convex functions.

Lemma 5.3 *An M$_2$-convex set is integrally-convex.*

Proof. For an M$_2$-convex set $S \subseteq \mathbf{Z}^V$ and any vectors $a, b \in \mathbf{Z}^V$ with $a \leq b$, the set $\overline{S} \cap [a, b]$ is an integral polyhedron (see, e.g., [5]). This implies (2.5), i.e, S is an integrally-convex set. □

Lemma 5.4 *An M-convex function* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *is integrally-convex.*

Proof. For $x \notin \overline{\mathrm{dom}\, f}$, we have $\overline{f}(x) = +\infty$, which, combined with (2.4), implies that $\widetilde{f}(x) = \overline{f}(x) = +\infty$.

To show $\overline{f}(x) = \widetilde{f}(x)$ for $x \in \overline{\mathrm{dom}\, f}$, we consider the following dual pair of linear programming problems:

$$\begin{array}{lll} \text{(LP1)} & \text{Maximize} & \langle p, x\rangle + \alpha \\ & \text{subject to} & \langle p, y\rangle + \alpha \leq f(y) \quad (y \in \mathrm{N}_1(x)), \;\; p \in \mathbf{R}^V, \;\; \alpha \in \mathbf{R}, \\ \text{(LP2)} & \text{Minimize} & \sum_{y \in \mathrm{N}_1(x)} \lambda_y f(y) \\ & \text{subject to} & \sum_{y \in \mathrm{N}_1(x)} \lambda_y y = x, \quad \sum_{y \in \mathrm{N}_1(x)} \lambda_y = 1, \; \lambda_y \geq 0 \quad (y \in \mathrm{N}_1(x)). \end{array}$$

By Lemma 5.3, we have $x \in \overline{\mathrm{N}_1(x) \cap \mathrm{dom}\, f}$. Hence, (LP2) has a feasible solution. Let (p^*, α^*) ($\in \mathbf{R}^V \times \mathbf{R}$) and $\lambda^* = (\lambda_y^* \mid y \in \mathrm{N}_1(x))$ be optimal solutions of (LP1) and (LP2), respectively. Then, it holds that

$$\overline{f}(x) \leq \langle p^*, x\rangle + \alpha^* = \sum_{y \in \mathrm{N}_1(x)} \lambda_y^* f(y) \leq \widetilde{f}(x). \tag{5.3}$$

We will show that both of the inequalities hold with equality.

Put

$$B = \{y \in \mathrm{N}_1(x) \mid \langle p^*, y\rangle + \alpha^* = f(y)\} = \arg\min_{y \in \mathrm{N}_1(x)} f[-p^*](y),$$

which is an M-convex set. The complementary slackness condition yields that $\{y \in \mathrm{N}_1(x) \mid \lambda_y^* > 0\} \subseteq B$, which implies $x \in \overline{B}$. In particular, we have $x \in \overline{B \cap \mathrm{N}_0(x)}$ by Lemma 5.3. Hence, there is another optimal solution $\widetilde{\lambda} = (\widetilde{\lambda}_y \mid y \in \mathrm{N}_1(x))$ of (LP2) such that if $\widetilde{\lambda}_y > 0$ then $y \in B \cap \mathrm{N}_0(x)$. Since

$$\sum_{y \in \mathrm{N}_1(x)} \lambda_y^* f(y) = \sum_{y \in \mathrm{N}_1(x)} \widetilde{\lambda}_y f(y) = \sum_{y \in \mathrm{N}_0(x)} \widetilde{\lambda}_y f(y) \geq \widetilde{f}(x),$$

the second inequality in (5.3) holds with equality.

Let $y_0 \in B \cap \mathrm{N}_0(x)$. Then, we have $f[-p^*](y_0 - \chi_u + \chi_v) \geq f[-p^*](y_0)$ for any $u, v \in V$. Since local optimality means global optimality for M-convex functions [11, Theorem 4.6], we have $\alpha^* = f[-p^*](y_0) \leq f[-p^*](y) = -\langle p^*, y\rangle + f(y)$ ($\forall y \in \mathrm{dom}\, f$), i.e., $\langle p^*, y\rangle + \alpha^* \leq f(y)$ ($\forall y \in \mathrm{dom}\, f$). By the equation (2.1) for $\overline{f}$, the first inequality in (5.3) holds with equality. □

The following theorem claims that we can choose a common optimal λ in (2.3) for two M-convex functions.

Lemma 5.5 *For two M-convex functions* $f, g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *and a vector* $x \in \mathbf{R}^V$*, there exists* $\lambda = (\lambda_y \mid y \in \mathrm{N}_0(x))$ *such that*

$$\sum_{y \in \mathrm{N}_0(x)} \lambda_y y = x, \qquad \sum_{y \in \mathrm{N}_0(x)} \lambda_y = 1, \qquad \lambda_y \geq 0 \;\; (y \in \mathrm{N}_0(x)), \tag{5.4}$$

and

$$\overline{f}(x) = \tilde{f}(x) = \sum_{y \in \mathrm{N}_0(x)} \lambda_y f(y), \qquad \overline{g}(x) = \tilde{g}(x) = \sum_{y \in \mathrm{N}_0(x)} \lambda_y g(y). \tag{5.5}$$

Proof. We may assume $x \in \overline{\mathrm{dom}\, f} \cap \overline{\mathrm{dom}\, g}$, which implies that both $\tilde{f}(x)$ and $\tilde{g}(x)$ are finite. By (2.2), there exist $(p, \alpha), (q, \beta) \in \mathbf{R}^V \times \mathbf{R}$ such that

$$\begin{array}{lll} \langle p, y \rangle + \alpha \leq f(y) & (y \in \mathrm{N}_0(x)), & \langle p, x \rangle + \alpha = \tilde{f}(x), \\ \langle q, y \rangle + \beta \leq g(y) & (y \in \mathrm{N}_0(x)), & \langle q, x \rangle + \beta = \tilde{g}(x). \end{array}$$

Put

$$B_f = \{y \in \mathrm{N}_0(x) \mid \langle p, y \rangle + \alpha = f(y)\}, \qquad B_g = \{y \in \mathrm{N}_0(x) \mid \langle q, y \rangle + \beta = g(y)\},$$

where both sets are M-convex. Then, we have $x \in \overline{B_f} \cap \overline{B_g} = \overline{B_f \cap B_g}$. Since $B_f \cap B_g$ is integrally-convex by Lemma 5.3, there exists $\lambda = (\lambda_y \mid y \in \mathrm{N}_0(x))$ satisfying (5.4) and $\lambda_y = 0$ $(y \notin B_f \cap B_g)$. Such λ satisfies (5.5) by the linear programming duality. □

We are now ready to prove that an M_2-convex function is integrally-convex. Let $f = f_1 + f_2$ be an M_2-convex function given as a sum of two M-convex functions $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. From Lemma 5.5 we have $\tilde{f}(x) = \tilde{f}_1(x) + \tilde{f}_2(x) = \overline{f}_1(x) + \overline{f}_2(x)$ $(x \in \mathbf{R}^V)$. Hence, $\tilde{f}$ is convex, i.e., f is integrally-convex.

5.3 Proof of Theorem 3.12

From the definition of $\mathrm{L}_2^\natural$-convex functions, it suffices to show that an L_2-convex function is integrally-convex. First of all, we consider the special case of L-convex functions. The convex closure of an L-convex function can be expressed explicitly as follows.

Theorem 5.6 ([11]) *An L-convex function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is integrally-convex; more specifically, for any $y \in \mathrm{dom}\, f$ and $a \in [0,1]^V$, we have*

$$\overline{f}(y + a) = (1 - \alpha_1) f(y) + \sum_{j=1}^{k-1} (\alpha_j - \alpha_{j+1}) f(y + \chi_{V_j}) + \alpha_k f(y + \chi_{V_k}), \tag{5.6}$$

$$\overline{f}(y - a) = (1 - \alpha_1) f(y) + \sum_{j=1}^{k-1} (\alpha_j - \alpha_{j+1}) f(y - \chi_{V_j}) + \alpha_k f(y - \chi_{V_k}), \tag{5.7}$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_k$ (≥ 0) are distinct values in $\{a(v)\}_{v \in V}$, and $V_j = \{v \in V \mid a(v) \geq \alpha_j\}$ $(j = 1, \cdots, k)$.

Proof. The equation (5.6) is shown in [11, Theorem 4.18], whereas (5.7) follows immediately from (5.6) since the function $f(-x)$ is L-convex in $x \in \mathbf{Z}^V$. □

We now prove that an L_2-convex function is integrally-convex. Let $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be L-convex functions. Since $\overline{f}_1 \Box \overline{f}_2$ is a convex function, it suffices to show $\widetilde{f} = \overline{f}_1 \Box \overline{f}_2$, which follows from the two claims below.

Claim 1 $(\overline{f}_1 \Box \overline{f}_2)(x) \leq \widetilde{f}(x)$ for any $x \in \mathbf{R}^V$.

Proof. If $\widetilde{f}(x) = +\infty$ then the inequality holds immediately. Hence, we assume $\widetilde{f}(x) < +\infty$. Let ε be any positive real number. Then, there exist vectors $y_{ij} \in \mathbf{Z}^V$ $(i = 1, 2; j = 1, 2, \cdots, m)$ and positive real values λ_j $(j = 1, \cdots, m)$ such that

$$y_{1j} + y_{2j} \in \mathrm{N}_0(x) \ (j = 1, \cdots, m), \qquad \sum_{j=1}^{m} \lambda_j = 1, \qquad \sum_{j=1}^{m} \lambda_j \{y_{1j} + y_{2j}\} = x,$$

$$(0 \leq) \ \sum_{j=1}^{m} \lambda_j \{f_1(y_{1j}) + f_2(y_{2j})\} - \widetilde{f}(x) \leq \varepsilon. \tag{5.8}$$

For $i = 1, 2$, put $x_i = \sum_{j=1}^{m} \lambda_j y_{ij}$. Then,

$$\overline{f}_i(x_i) \leq \sum_{j=1}^{m} \lambda_j f_i(y_{ij}) \qquad (i = 1, 2). \tag{5.9}$$

Since $x_1 + x_2 = x$, we have

$$(\overline{f}_1 \Box \overline{f}_2)(x) \leq \overline{f}_1(x_1) + \overline{f}_2(x_2). \tag{5.10}$$

Combining the inequalities (5.8), (5.9), and (5.10), we have $(\overline{f}_1 \Box \overline{f}_2)(x) - \widetilde{f}(x) \leq \varepsilon$, from which the claim follows since ε can be chosen arbitrarily. $\Box$

Claim 2 $\widetilde{f}(x) \leq (\overline{f}_1 \Box \overline{f}_2)(x)$ for any $x \in \mathbf{R}^V$.

Proof. It suffices to show that

$$\widetilde{f}(x) \leq \overline{f}_1(x_1) + \overline{f}_2(x_2) \tag{5.11}$$

holds for any $x_i \in \mathrm{dom}\, \overline{f}_i$ $(i = 1, 2)$ with $x_1 + x_2 = x$.

Put $a_1 = x_1 - \lfloor x_1 \rfloor$, $a_2 = \lceil x_2 \rceil - x_2$. Note that $0 \leq a_i(v) < 1$ $(i = 1, 2; v \in V)$. Let $\alpha_1 > \alpha_2 > \cdots > \alpha_k$ (≥ 0) be the distinct values in $\{a_1(v), a_2(v) \mid v \in V\}$, and put $V_{ij} = \{v \in V \mid a_i(v) \geq \alpha_j\}$ $(i = 1, 2; j = 1, \cdots, k)$. Then, we have

$$a_i = \sum_{j=1}^{k-1} (\alpha_j - \alpha_{j+1}) \chi_{V_{ij}} + \alpha_k \chi_{V_{ik}} \qquad (i = 1, 2). \tag{5.12}$$

Theorem 5.6 implies that

$$\overline{f}_1(x_1) = (1 - \alpha_1) f_1(\lfloor x_1 \rfloor) + \sum_{j=1}^{k-1} (\alpha_j - \alpha_{j+1}) f_1(\lfloor x_1 \rfloor + \chi_{V_{1j}}) + \alpha_k f_1(\lfloor x_1 \rfloor + \chi_{V_{1k}}), \tag{5.13}$$

$$\overline{f}_2(x_2) = (1 - \alpha_1) f_2(\lceil x_2 \rceil) + \sum_{j=1}^{k-1} (\alpha_j - \alpha_{j+1}) f_2(\lceil x_2 \rceil - \chi_{V_{2j}}) + \alpha_k f_2(\lceil x_2 \rceil - \chi_{V_{2k}}). \tag{5.14}$$

It follows from (5.12), (5.13) and (5.14) that

$$
\begin{aligned}
x_1 + x_2 &= (1-\alpha_1)\{\lfloor x_1 \rfloor + \lceil x_2 \rceil\} + \sum_{j=1}^{k-1}(\alpha_j - \alpha_{j+1})\{\lfloor x_1 \rfloor + \chi_{V_{1j}} + \lceil x_2 \rceil - \chi_{V_{2j}}\} \\
&\qquad + \alpha_k\{\lfloor x_1 \rfloor + \chi_{V_{1k}} + \lceil x_2 \rceil - \chi_{V_{2k}}\}, \qquad (5.15) \\
\overline{f}_1(x_1) + \overline{f}_2(x_2) &= (1-\alpha_1)\{f_1(\lfloor x_1 \rfloor) + f_2(\lceil x_2 \rceil)\} \\
&\qquad + \sum_{j=1}^{k-1}(\alpha_j - \alpha_{j+1})\{f_1(\lfloor x_1 \rfloor + \chi_{V_{1j}}) + f_2(\lceil x_2 \rceil - \chi_{V_{2j}})\} \\
&\qquad + \alpha_k\{f_1(\lfloor x_1 \rfloor + \chi_{V_{1k}}) + f_2(\lceil x_2 \rceil - \chi_{V_{2k}})\} \\
&\geq (1-\alpha_1)f(\lfloor x_1 \rfloor + \lceil x_2 \rceil) + \sum_{j=1}^{k-1}(\alpha_j - \alpha_{j+1})f(\lfloor x_1 \rfloor + \chi_{V_{1j}} + \lceil x_2 \rceil - \chi_{V_{2j}}) \\
&\qquad + \alpha_k f(\lfloor x_1 \rfloor + \chi_{V_{1k}} + \lceil x_2 \rceil - \chi_{V_{2k}}). \qquad (5.16)
\end{aligned}
$$

As shown below, we have

$$
\lfloor x_1 \rfloor + \lceil x_2 \rceil \in \mathrm{N}_0(x), \qquad (5.17)
$$

$$
\lfloor x_1 \rfloor + \chi_{V_{1j}} + \lceil x_2 \rceil - \chi_{V_{2j}} \in \mathrm{N}_0(x) \qquad (j = 1, 2, \cdots, k), \qquad (5.18)
$$

which, together with (5.15) and (5.16), imply the desired inequality (5.11).

To conclude the proof, we show (5.17) and (5.18). It follows from

$$
\begin{aligned}
&\lfloor x_1 \rfloor + \lceil x_2 \rceil = x - a_1 + a_2 \in \mathbf{Z}^V, \\
&\lfloor x(v) \rfloor \leq x(v) < \lfloor x(v) \rfloor + 1, \quad -1 < -a_1(v) + a_2(v) < 1 \qquad (v \in V),
\end{aligned}
$$

that for any $v \in V$ the value $\lfloor x_1(v) \rfloor + \lceil x_2(v) \rceil$ is equal to $x(v)$ if $x(v) \in \mathbf{Z}$, and equal to either $\lfloor x(v) \rfloor$ or $\lfloor x(v) \rfloor + 1$ if $x(v) \notin \mathbf{Z}$. Hence, we have (5.17).

Put

$$
W = \{v \in V \mid \lfloor x_1(v) \rfloor + \lceil x_2(v) \rceil = \lfloor x(v) \rfloor + 1\}.
$$

It holds that $x(v) \notin \mathbf{Z}$ for $v \in W$ and

$$
\lfloor x_1 \rfloor + \chi_{V_{1j}} + \lceil x_2 \rceil - \chi_{V_{2j}} = \lfloor x \rfloor + \chi_W + \chi_{V_{1j}} - \chi_{V_{2j}}.
$$

To prove (5.18), it suffices to show that

(i) if $x(v) \in \mathbf{Z}$ then $v \in V_{1j} \cap V_{2j}$ or $v \in V - (V_{1j} \cup V_{2j})$,
(ii) if $v \in W \cap V_{1j}$, then $v \in V_{2j}$,
(iii) if $v \in V_{2j} - W$, then $v \in V_{1j}$.

(i) If $x(v) \in \mathbf{Z}$, then we have $a_1(v) = a_2(v) = \alpha_j$ for some j, from which (i) follows.

(ii) If $v \in W \cap V_{1j}$, then $v \in V_{2j}$ since

$$
a_2(v) = \lfloor x_1(v) \rfloor + \lceil x_2(v) \rceil - x(v) + a_1(v) = \lfloor x(v) \rfloor + 1 - x(v) + a_1(v) \geq a_1(v) \geq \alpha_j.
$$

(iii) If $v \in V_{2j} - W$, then $v \in V_{1j}$ since

$$
a_1(v) = -\lfloor x_1(v) \rfloor - \lceil x_2(v) \rceil + x(v) + a_2(v) = -\lfloor x(v) \rfloor + x(v) + a_2(v) \geq a_2(v) \geq \alpha_j.
$$

□

5.4 Proof of Theorem 3.17

The proof is based on the following fact.

Lemma 5.7 *A set $S \subseteq \mathbf{Z}^V$ is separable-convex if and only if S is both $\mathrm{M}_2^\natural$-convex and $\mathrm{L}_2^\natural$-convex.*

Proof. We show the "if" part only. For each $v \in V$, put $a(v) = \inf_{x \in S} x(v)$ and $b(v) = \sup_{x \in S} x(v)$. Obviously, $S \subseteq [a, b]$ holds. In the following, we prove $[a, b] \subseteq S$.

Let $S_1, S_2 \subseteq \mathbf{Z}^V$ be $\mathrm{L}^\natural$-convex sets such that $S = S_1 + S_2$. Let $x \in [a, b]$. Then, for each $v \in V$ there exist vectors $p_v, q_v \in S$ such that $p_v(v) \leq x(v) \leq q_v(v)$. Moreover, there exist vectors $p_{vi}, q_{vi} \in S_i$ $(i = 1, 2)$ such that $p_{v1} + p_{v2} = p_v$, $q_{v1} + q_{v2} = q_v$. Put

$$p_i = \bigwedge_{v \in V} p_{vi} \in S_i \quad (i = 1, 2), \qquad p = p_1 + p_2 \in S,$$
$$q_i = \bigvee_{v \in V} q_{vi} \in S_i \quad (i = 1, 2), \qquad q = q_1 + q_2 \in S.$$

Then, we have

$$p(v) = p_1(v) + p_2(v) \leq p_{v1}(v) + p_{v2}(v) = p_v(v) \leq x(v) \qquad (v \in V),$$
$$q(v) = q_1(v) + q_2(v) \geq q_{v1}(v) + q_{v2}(v) = q_v(v) \geq x(v) \qquad (v \in V),$$

i.e., $x \in [p, q]$. Hence, we have $x \in [p, q] \subseteq S$ by Lemma 5.2 and the $\mathrm{M}_2^\natural$-convexity of S. This shows that $[a, b] \subseteq S$. □

We now prove the "if" part of Theorem 3.17. The "only if" part is obvious from Lemma 3.16.

Assume that $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is both $\mathrm{M}_2^\natural$-convex and $\mathrm{L}_2^\natural$-convex. Then, f satisfies the following properties:

(i) f is integrally convex (by Theorems 3.9, 3.12),
(ii) $\mathrm{dom}\, f$ is a separable convex set (by Theorems 4.25 (i), 4.26 (i), Lemma 5.7),
(iii) for any $p \in \mathbf{R}^V$, $\arg\min f[-p]$ is a separable-convex set (by Theorems 4.25 (ii), 4.26 (ii), Lemma 5.7).

Due to the property (iii), the function f is linear over each hypercube $[x', x' + \mathbf{1}]$ $(x' \in \mathbf{Z}^V)$, which implies that $f(x + \chi_v) - f(x) = f(y + \chi_v) - f(y)$ for any $x, y \in \mathrm{dom}\, f$ with $x(v) = y(v)$. For all $v \in V$, put

$$f_v(\alpha) = f(x_0 + (\alpha - x_0(v))\chi_v) - f(x_0) \qquad (\alpha \in \mathbf{Z}),$$

where $x_0 \in \mathrm{dom}\, f$. Then, we have $f(x) = \sum_{v \in V} f_v(x(v)) + f(x_0)$ $(x \in \mathbf{Z}^V)$. Moreover, each f_v is convex since f is integrally-convex. Therefore, f is a separable-convex function.

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