A Constructive Proof for the Induction of M-convex Functions through Networks

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Abstract

Murota (1995) introduced an M-convex function as a quantitative generalization of the set of integral vectors in an integral base polyhedron as well as an extension of valuated matroid over base polyhedron. Just as a base polyhedron can be transformed through a network, an M-convex function can be induced through a network. This paper gives a constructive proof for the induction of an M-convex function. The proof is based on the correctness of a simple algorithm, which finds an exchangeable element. We also analyze a behavior of induced functions when they take the value $-\infty$.

Keywords: matroid, base polyhedron, submodular system, convex function.

1 Introduction

In 1990, Dress–Wenzel introduced a valuated matroid as a quantitative generalization of a matroid [1, 2]. A valuated matroid is a pair of a matroid $(E, \mathcal{B})$ and a function $\omega : \mathcal{B} \to \mathbf{R}$ which enjoys the following exchange property:

(VM) $\forall X, Y \in \mathcal{B}, \forall u \in X - Y, \exists v \in Y - X$ such that $X - u + v \in \mathcal{B}, Y + u - v \in \mathcal{B}$, and

$$\omega(X) + \omega(Y) \leq \omega(X - u + v) + \omega(Y + u - v).$$

Such a function ω is called a valuation of $(E, \mathcal{B})$.

Recently, Murota introduced the concept of M-convex function [5, 6, 7], which is a quantitative generalization of integral vectors in an integral base polyhedron as well as an extension of (the negative of) matroid valuation over base polyhedron. It is known that the set of integral vectors in an integral base polyhedron $B \subseteq \mathbf{Z}^E$ is characterized by the exchange property

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(B-EXC) $\forall x, y \in B$, $\forall u \in E$ with $x(u) > y(u)$, $\exists v \in E$ with $x(v) < y(v)$ such that $x - \chi_u + \chi_v \in B$, $y + \chi_u - \chi_v \in B$,

where $\chi_u \in \{0,1\}^E$ is the characteristic vector of $u \in E$. In contrast, an M-convex function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{+\infty\}$ satisfies, by definition, the following quantitative generalization of the simultaneous exchange property:

(M-EXC) $\forall x, y \in B$, $\forall u \in E$ with $x(u) > y(u)$, $\exists v \in E$ with $x(v) < y(v)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

Here dom g denotes the set $\{x \in \mathbf{Z}^E \mid g(x) < +\infty\}$ for $g : \mathbf{Z}^E \to \mathbf{R} \cup \{\pm\infty\}$. The property (M-EXC) implies (B-EXC) for dom f of such f. Note that a matroid valuation is nothing but an M-concave (the negative of M-convex) function f with dom $f \subseteq \{0,1\}^E$.

The framework of M-convex function gives us a new understanding for the well-solvability of nonlinear discrete optimization problems, e.g., the convex cost flow problem, the nonlinear resource allocation problem (see [4, 5, 6, 7]). An M-convex function enjoys nice properties such as the extendability to an ordinary convex function, the success of Fenchel-type duality and a (discrete) separation theorem, which convince us that the name "M-convex" is reasonable.

In the theory of (poly-)matroid, there have been considered several operations such as reduction, contraction, etc. (see [3] as a relevant reference). Above all, the induction by networks is one of the most powerful operations for matroids and base polyhedra and includes other operations as special cases. Recent works by Murota [5, 7] revealed that the network induction also applies to M-convex functions.

Let $G = (V, A; V^+, V^-)$ be a directed graph with two specified vertex sets $V^+, V^- \subseteq V$ such that $V^+ \cap V^- = \emptyset$. We denote an upper and lower capacity functions by $\overline{c} : A \to \mathbf{Z} \cup \{+\infty\}$, $\underline{c} : A \to \mathbf{Z} \cup \{-\infty\}$, respectively, and a weight function by $\gamma : A \to \mathbf{R}$. For any function $z : V \to \mathbf{R}$, we denote the restriction of z to V^+ and to V^- by $(z)^+$ and $(z)^-$, respectively. Let $f^+ : \mathbf{Z}^{V^+} \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function. For any flow $\varphi : A \to \mathbf{Z}$, its boundary $\partial\varphi : V \to \mathbf{Z}$ is defined as

$$\partial\varphi(v) = \sum\{\varphi(a) \mid a(\in A) \text{ leaves } v\} - \sum\{\varphi(a) \mid a(\in A) \text{ enters } v\} \qquad (v \in V),$$

and its cost is $\mathrm{cost}(\varphi) = \sum\{\gamma(a)\varphi(a) \mid a \in A\} + f^+((\varphi)^+)$. A flow φ is called feasible if it satisfies the following conditions:

$$\underline{c}(a) \leq \varphi(a) \leq \overline{c}(a) \ (a \in A), \quad \partial\varphi(v) = 0 \ (v \in V - (V^+ \cup V^-)), \quad (\partial\varphi)^+ \in \mathrm{dom}\, f^+.$$

We define a function $f : \mathbf{Z}^{V^-} \to \mathbf{R} \cup \{\pm\infty\}$ as follows:

$$f(x) = \begin{cases} \inf\{\mathrm{cost}(\varphi) \mid \varphi : \text{feasible flow}, (\partial\varphi)^- = x\} & (\exists \text{ feasible flow } \varphi \text{ with } (\partial\varphi)^- = x), \\ +\infty & (\text{otherwise}). \end{cases}$$

We have the following theorem, which is proved by Murota [5, Theorem 7.2], [7, Theorem 4.14] based on a characterization of M-convexity by minimizers and on an optimality condition for the generalized submodular flow problem [6].

Theorem 1.1 *The function* f *is M-convex if it does not take the value* $-\infty$.

The objective of this paper is to provide an alternative simpler proof of this theorem. Our proof is fairly straightforward and constructive, by establishing directly the condition (M-EXC) for the induced function f. The essence of the proof lies in the correctness of a simple algorithm, which for any $x, y \in \mathrm{dom}\, f$ and $u \in V^-$ with $x(u) > y(u)$, finds a vertex $v \in V^-$ with $x(v) < y(v)$ such that $f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v) \leq f(x) + f(y)$. More specifically, our algorithm INDUCTION has the following property:

Theorem 1.2 *Given feasible flows* φ, ψ *and* $u \in V^-$ *with* $\partial\varphi(u) > \partial\psi(u)$, *the algorithm* INDUCTION *finds feasible flows* φ', ψ' *and* $v \in V^-$ *with* $\partial\varphi(v) < \partial\psi(v)$ *satisfying* $(\partial\varphi')^- = (\partial\varphi)^- - \chi_u + \chi_v$, $(\partial\psi')^- = (\partial\psi)^- + \chi_u - \chi_v$, *and*

$$\mathrm{cost}(\varphi') + \mathrm{cost}(\psi') \leq \mathrm{cost}(\varphi) + \mathrm{cost}(\psi).$$

The proof of this theorem is given in Section 3.

We also analyze a behavior of induced functions when they take the value $-\infty$. Compared with the case where f does not take $-\infty$, a behavior of f is not known yet when it takes $-\infty$. Exploiting Theorem 1.2 we show that such f takes $-\infty$ for any 'interior' vector of dom f.

2 The Proof for the Induction

Based on Theorem 1.2, we first assert a slightly stronger claim than Theorem 1.1. Note that the induced function f may take the value $-\infty$ while an M-convex function does not by definition.

Theorem 2.1 *For any* $x, y \in \mathrm{dom}\, f$ *and* $u \in V^-$ *with* $x(u) > y(u)$, *there exists* $v \in V^-$ *with* $x(v) < y(v)$ *such that* $x - \chi_u + \chi_v \in \mathrm{dom}\, f$, $y + \chi_u - \chi_v \in \mathrm{dom}\, f$, *and*

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

Proof. Let $\{\varphi_k\}_{k=1}^{\infty}$, $\{\psi_k\}_{k=1}^{\infty}$ be sequences of feasible flows with $(\partial\varphi_k)^- = x$, $(\partial\psi_k)^- = y$ such that $\lim_{k\to\infty} \mathrm{cost}(\varphi_k) = f(x)$, $\lim_{k\to\infty} \mathrm{cost}(\psi_k) = f(y)$. For each k, Theorem 1.2 assures the existence of feasible flows φ'_k, ψ'_k and a vertex $v_k \in V^-$ with $x(v_k) < y(v_k)$ such that $(\partial\varphi'_k)^- = x - \chi_u + \chi_{v_k}$, $(\partial\psi'_k)^- = y + \chi_u - \chi_{v_k}$, and $\mathrm{cost}(\varphi'_k) + \mathrm{cost}(\psi'_k) \leq \mathrm{cost}(\varphi_k) + \mathrm{cost}(\psi_k)$. Since the vertex set V^- is finite, there is at least one vertex v appearing infinitely in the sequence $\{v_k\}_{k=1}^{\infty}$. Thus, we have an inequality

$$\begin{aligned} f(x) + f(y) &= \inf_{k:v_k=v} \mathrm{cost}(\varphi_k) + \inf_{k:v_k=v} \mathrm{cost}(\psi_k) \\ &\geq \inf_{k:v_k=v} \mathrm{cost}(\varphi'_k) + \inf_{k:v_k=v} \mathrm{cost}(\psi'_k) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v). \quad \blacksquare \end{aligned}$$

Theorem 2.1 reads as follows if $f(x) = -\infty$ or $f(y) = -\infty$:

Corollary 2.2 *Let* $x, y \in \mathrm{dom}\, f$ *with either* $f(x) = -\infty$ *or* $f(y) = -\infty$. *Then, for any* $u \in V^-$ *with* $x(u) > y(u)$, *there exists* $v \in V^-$ *with* $x(v) < y(v)$ *such that either* $f(x - \chi_u + \chi_v) = -\infty$ *or* $f(y + \chi_u - \chi_v) = -\infty$.

On the other hand, the claim of Theorem 2.1 is just (M-EXC) if f does not take $-\infty$.

Exploiting Corollary 2.2, we reveal that $f(x) = -\infty$ for any vector x in the 'interior' of $\mathrm{dom}\, f$ if f takes $-\infty$. Since $\mathrm{dom}\, f$ fulfills (B-EXC) by Theorem 2.1, there exists a submodular function $\rho : 2^{V^-} \to \mathbf{Z} \cup \{+\infty\}$ such that

$$\mathrm{dom}\, f = \{x \in \mathbf{Z}^{V^-} \mid x(X) \le \rho(X) \ (\forall X \subseteq V^-),\ x(V^-) = \rho(V^-)\}.$$

For any vector $x \in \mathbf{Z}^E$, denote by $||x||$ the value $\sum\{x(w) \mid w \in E\}$. Compare the next theorem with the related result for the convolution operation [7, Theorem 5.8 (2)], which is a special case of the network induction [5].

Theorem 2.3 *Let* $x_0 \in \mathrm{dom}\, f$ *with* $f(x_0) = -\infty$. *Then* $f(x) = -\infty$ *for all* $x \in I(x_0)$, *where*

$$I(x_0) = \{x \in \mathrm{dom}\, f \mid \forall X \subseteq V^-,\ x_0(X) = \rho(X) \text{ if } x(X) = \rho(X)\}.$$

Proof. We show by induction on the value $||x - x_0||$.

Suppose $x = x_0 - \chi_s + \chi_t$ for $s, t \in V^-$, $s \neq t$. Let $x' = x - \chi_s + \chi_t$. For any $X \subseteq V^-$, if $x(X) = \rho(X)$ and $t \in X$ then $s \in X$ since $x_0(X) = x(X)$. Hence $x'(X) = \rho(X)$, which concludes $x' \in \mathrm{dom}\, f$. Applying Corollary 2.2 to x_0 and x', we have $f(x) = -\infty$.

Next, assume $||x - x_0|| \ge 4$. Apply Corollary 2.2 to x_0 and x and obtain either $f(x_0 - \chi_u + \chi_v) = -\infty$ or $f(x + \chi_u - \chi_v) = -\infty$ for $u, v \in V^-$ with $x_0(u) > x(u)$ and $x_0(v) < x(v)$. Moreover, if $x(X) = \rho(X) (= x_0(X))$ then $(x_0 - \chi_u + \chi_v)(X) = (x + \chi_u - \chi_v)(X) = \rho(X)$, yielding $x \in I(x_0 - \chi_u + \chi_v) \cap I(x + \chi_u - \chi_v)$. The assumption of the induction implies $f(x) = -\infty$ since $||x - (x_0 - \chi_u + \chi_v)|| = ||x - x_0|| - 2$ and $||x - (x + \chi_u - \chi_v)|| = 2$. ∎

3 An Algorithm

This section proves Theorem 1.2 by showing the algorithm Induction. Assume w.l.o.g. $V^+ \cup V^- = V$, otherwise extend the function $f^+ : \mathbf{Z}^{V^+} \to \mathbf{R} \cup \{+\infty\}$ over $\mathbf{Z}^{V - V^-}$ as

$$\overline{f}^+(x^+, x^0) = \begin{cases} f^+(x^+) & (x^0 = \mathbf{0}), \\ +\infty & (x^0 \neq \mathbf{0}), \end{cases} \qquad (x^+ \in \mathbf{Z}^{V^+}, x^0 \in \mathbf{Z}^{V - (V^+ \cup V^-)}),$$

and reset V^+ to $V - V^-$.

Input of the algorithm is feasible flows φ, ψ and a vertex $u \in V^-$ with $\partial\varphi(u) > \partial\psi(u)$. The algorithm maintains a set of four functions $\varphi', \psi' \in \mathbf{Z}^A,\ b', d' \in \mathbf{Z}^V$ and a vertex $w \in V$ satisfying the following condition (FBS):

$$\textbf{(FBS)} \begin{cases} \underline{c}(a) \le \varphi'(a) \le \overline{c}(a),\ \underline{c}(a) \le \psi'(a) \le \overline{c}(a) \quad (a \in A), \\ (b')^+,\ (d')^+ \in \mathrm{dom}\, f^+,\ (b')^- = (\partial\varphi)^- - \chi_u,\ (d')^- = (\partial\psi)^- + \chi_u, \\ b' = \partial\varphi' - \chi_w,\ d' = \partial\psi' + \chi_w, \\ F^+(\varphi', b') + F^+(\psi', d') \le F^+(\varphi, \partial\varphi) + F^+(\psi, \partial\psi), \end{cases}$$

where $F^+(\varphi', b') = \sum\{\gamma(a)\varphi'(a) \mid a \in A\} + f^+((b')^+)$. We refer to such a tuple $(\varphi', \psi', b', d', w)$ as a *flow-base set.* Note that b' and d' of some flow-base set $(\varphi', \psi', b', d', w)$ are uniquely determined by φ', ψ', and w. Our aim is to obtain a flow-base set $(\varphi', \psi', b', d', v)$ such that $v \in V^-$ and $\partial\varphi(v) < \partial\psi(v)$ since the flows φ', ψ' satisfy the condition of Theorem 1.2.

Algorithm INDUCTION

Step 0: Put $k = 1$, $(\varphi_1, \psi_1, b_1, d_1, v_1) = (\varphi, \psi, \partial\varphi - \chi_u, \partial\psi + \chi_u, u)$.

Step 1: Let $(\varphi', \psi', b', d', v_k)$ be a flow-base set with v_k fixed which minimizes the value $||\varphi' - \psi'||$. Set $(\varphi'_k, \psi'_k, b'_k, d'_k, v_k)$ as $(\varphi', \psi', b', d', v_k)$ if $||\varphi' - \psi'|| < ||\varphi_k - \psi_k||$, and $(\varphi_k, \psi_k, b_k, d_k, v_k)$ otherwise.

Step 2: CASE 1: If $v_k \in V^-$ and $\partial\varphi(v_k) < \partial\psi(v_k)$, output $(\varphi'_k, \psi'_k, b'_k, d'_k, v_k)$ and stop.
CASE 2: If Case 1 does not happen and $L_k \cup E_k - \{a_{k-1}$ (if defined)$\} \neq \emptyset$, where

$$L_k = \{a \in A \mid a \text{ leaves } v_k,\ \varphi'_k(a) > \psi'_k(a)\}, \quad E_k = \{a \in A \mid a \text{ enters } v_k,\ \varphi'_k(a) < \psi'_k(a)\},$$

then take any arc $a_k \in L_k \cup E_k - \{a_{k-1}$ (if defined)$\}$. Let v_{k+1} be another end vertex of a_k. Set

$$\varphi_{k+1}(a_k) = \begin{cases} \varphi'_k(a_k) - 1 & (\text{if } a_k \in L_k), \\ \varphi'_k(a_k) + 1 & (\text{if } a_k \in E_k), \end{cases} \quad \psi_{k+1}(a_k) = \begin{cases} \psi'_k(a_k) + 1 & (\text{if } a_k \in L_k), \\ \psi'_k(a_k) - 1 & (\text{if } a_k \in E_k). \end{cases}$$

Put $\varphi_{k+1}(a) = \varphi'_k(a)$, $\psi_{k+1}(a) = \psi'_k(a)$ $(\forall a \in A - a_k)$, $b_{k+1} = b'_k, d_{k+1} = d'_k$.
CASE 3: If neither Case 1 nor 2 holds and v_k fulfills $v_k \in V^+$ and $b'_k(v_k) < d'_k(v_k)$, then find a vertex $v_{k+1} \in V^+$ with $b'_k(v_{k+1}) > d'_k(v_{k+1})$ such that

$$f^+((b'_k)^+ + \chi_{v_k} - \chi_{v_{k+1}}) + f^+((d'_k)^+ - \chi_{v_k} + \chi_{v_{k+1}}) \le f^+((b'_k)^+) + f^+((d'_k)^+).$$

Put $(\varphi_{k+1}, \psi_{k+1}, b_{k+1}, d_{k+1}, v_{k+1}) = (\varphi'_k, \psi'_k, b'_k + \chi_{v_k} - \chi_{v_{k+1}}, d'_k - \chi_{v_k} + \chi_{v_{k+1}}, v_{k+1})$.

Step 3: Set $k = k + 1$ and go to Step 1. [End of Algorithm]

To the end of this section we prove the correctness of the algorithm. Suppose the algorithm runs correctly until the $(k-1)$-st iteration $(k \ge 1)$. It may be obvious that $(\varphi_k, \psi_k, b_k, d_k, v_k)$ is a flow-base set.

Lemma 3.1 *If Case 3 occurs in the $(k-1)$-st iteration and $(\varphi'_k, \psi'_k, b'_k, d'_k, v_k) = (\varphi_k, \psi_k, b_k, d_k, v_k)$, then Case 3 does not occur in the k-th iteration.*

Proof. By assumption $b'_{k-1}(v_k) \geq d'_{k-1}(v_k) + 1$, and therefore $b'_k(v_k) \geq d'_k(v_k) - 1$. Thus,

$$\begin{aligned} &\sum\{\varphi'_k(a) - \psi'_k(a) \mid a \text{ leaves } v_k\} - \sum\{\varphi'_k(a) - \psi'_k(a) \mid a \text{ enters } v_k\} \\ &\quad = \partial\varphi'_k(v_k) - \partial\psi'_k(v_k) \;=\; (b'_k(v_k) + 1) - (d'_k(v_k) - 1) \;\geq\; 1. \end{aligned}$$

This means $L_k \cup E_k \neq \emptyset$ and Case 3 does not appear. ∎

Lemma 3.2 $v_k \notin \{v_1, \cdots, v_{k-1}\}$.

Proof. Assume to the contrary that $v_k \in \{v_1, \cdots, v_{k-1}\}$ and let $i(< k)$ be the largest integer with $v_i = v_k$. Note that $i \leq k-2$ and that $v_i, v_{i+1}, \cdots, v_{k-1}$ are distinct. Since

$$F^+(\varphi'_k, b'_k) + F^+(\psi'_i, d'_i) + F^+(\varphi'_i, b'_i) + F^+(\psi'_k, d'_k) \leq 2\{F^+(\varphi, \partial\varphi) + F^+(\psi, \partial\psi)\},$$

either $(\varphi'_k, \psi'_i, b'_k, d'_i, v_i)$ or $(\varphi'_i, \psi'_k, b'_i, d'_k, v_i)$ is a flow-base set. In the following, we derive a contradiction by showing $||\varphi'_k - \psi'_i|| < ||\varphi'_i - \psi'_i||$ and $||\varphi'_i - \psi'_k|| < ||\varphi'_i - \psi'_i||$. Since $||\varphi'_i - \psi'_i|| = ||\varphi'_k - \psi'_k||$ by the setting in Step 1, it holds

$$||\varphi'_i - \psi'_i|| = ||\varphi_{i+1} - \psi_{i+1}|| = ||\varphi'_{i+1} - \psi'_{i+1}|| = \cdots = ||\varphi'_k - \psi'_k||.$$

Therefore, for any j $(i+1 \leq j \leq k)$, we have $\varphi'_j = \varphi_j$, $\psi'_j = \psi_j$ and

$$||\varphi'_j - \psi'_i|| = \begin{cases} ||\varphi'_{j-1} - \psi'_i|| - 1 & \text{(Case 2)}, \\ ||\varphi'_{j-1} - \psi'_i|| & \text{(Case 3)}, \end{cases} \quad ||\varphi'_i - \psi'_j|| = \begin{cases} ||\varphi'_i - \psi'_{j-1}|| - 1 & \text{(Case 2)}, \\ ||\varphi'_i - \psi'_{j-1}|| & \text{(Case 3)}. \end{cases}$$

Lemma 3.1 yields that $||\varphi'_k - \psi'_i|| < ||\varphi'_i - \psi'_i||$, $||\varphi'_i - \psi'_k|| < ||\varphi'_i - \psi'_i||$ since $i \leq k-2$. ∎

Lemma 3.3 *If neither Case 1 nor 2 happens, then v_k satisfies $v_k \in V^+$ and $b'_k(v_k) < d'_k(v_k)$.*

Proof. $L_k \cup E_k \subseteq \{a_{k-1}$ (if defined)$\}$ holds since Case 2 does not happen, and $L_k \cup E_k = \{a_{k-1}\}$ if and only if Case 2 happens in the $(k-1)$-st iteration and $|\varphi'_{k-1}(a_{k-1}) - \psi'_{k-1}(a_{k-1})| = 1$. Thus,

$$\partial\varphi'_k(v_k) - \partial\psi'_k(v_k) = \sum\{\varphi'_k(a) - \psi'_k(a) \mid a \text{ leaves } v_k\} - \sum\{\varphi'_k(a) - \psi'_k(a) \mid a \text{ enters } v_k\} \leq 1,$$

which provides $b'_k(v_k) < d'_k(v_k)$ since $(\varphi'_k, \psi'_k, b'_k, d'_k, v_k)$ satisfies (FBS). We also have $(b'_k)^- - (d'_k)^- = (\partial\varphi)^- - (\partial\psi)^- - 2\chi_u$, from which it follows

$$v_k \in \{w \in V \mid b'_k(w) < d'_k(w)\} \subseteq V^+ \cup \{w \in V^- \mid \partial\varphi(w) < \partial\psi(w)\} \cup \{u\}.$$

Since Case 1 does not happen, we have only to show $v_k \neq u$. Assume $v_k = u$. Then, $k = 1$ by Lemma 3.2 and $L_1 \cup E_1 \neq \emptyset$ since $\partial\varphi'_1(v_1) = \partial\varphi(u) > \partial\psi(u) = \partial\psi'_1(v_1)$. Hence, Case 2 happens necessarily, a contradiction. ∎

The above lemmas imply that the algorithm necessarily terminates in finite iterations and outputs the desired flow-base set, which concludes the proof of Theorem 1.2.

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