Neighbor Systems, Jump Systems, and Bisubmodular Polyhedra

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Abstract. The concept of neighbor system, introduced by Hartvigsen (2009), is a set of integral vectors satisfying a certain combinatorial property. In this paper, we reveal the relationship of neighbor systems with jump systems and with bisubmodular polyhedra. We firstly prove that for every neighbor system, there exists a jump system which has the same neighborhood structure as the original neighbor system. This statement shows that the concept of neighbor system is essentially equivalent to that of jump system. We then show that the convex closure of a neighbor system is an integral bisubmodular polyhedron. In addition, we give a characterization of neighbor systems using bisubmodular polyhedra. Finally, we consider the problem of minimizing a separable convex function on a neighbor system. By using the relationship between neighbor systems and jump systems shown in this paper, we prove that the problem can be solved in weakly-polynomial time for a class of neighbor systems.

1 Introduction

The concept of neighbor system, introduced by Hartvigsen [14], is a set of integral vectors satisfying a certain combinatorial property. The definition of neighbor system is as follows. Throughout this paper, let E be a finite set with n elements. Let $\mathcal{F}$ be a set of integral vectors in $\mathbf{Z}^E$. For $x, y \in \mathcal{F}$, we say that y is a *neighbor* of x if there exist some vector $d \in \{0, +1, -1\}^E$ with $|\{e \in E \mid d(e) \neq 0\}| \leq 2$ and a positive integer α such that $y = x + \alpha d$ and $x + \alpha' d \notin \mathcal{F}$ for all $\alpha' \in \mathbf{Z}$ with $0 < \alpha' < \alpha$. The set $\mathcal{F}$ is called an *(all-)neighbor system* if it satisfies the following axiom:

for every $x, y \in \mathcal{F}$ and $i \in E$ with $x(i) \neq y(i)$, there exists a neighbor $z \in \mathcal{F}$ of x such that $\min\{x(e), y(e)\} \leq z(e) \leq \max\{x(e), y(e)\}$ $(\forall e \in E)$ and $z(i) \neq x(i)$.

See Fig. 1 for an example of a 2-dimensional neighbor system. Given a positive integer k, a neighbor system $\mathcal{F}$ is said to be an N_k*-neighbor system* if we can always choose a neighbor z in the axiom above such that $||z - x||_1 \leq k$. For example, the neighbor system in Fig. 1 is an N_k-neighbor system for every $k \geq 3$, but not for $k = 1, 2$ since if $x = (0, 2)$ and $y = (3, 5)$ then we do not have such a neighbor z with $||z - x||_1 \leq 2$.

Neighbor system is a common generalization of various concepts such as matroid, integral polymatroid, delta-matroid, integral bisubmodular polyhedron, and jump systems. Below we review these concepts; see [12] for more accounts.

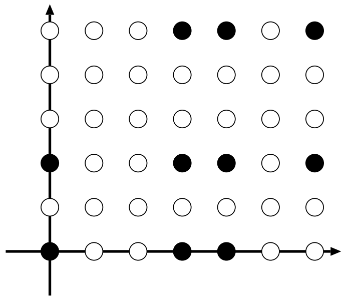

Fig. 1. An example of 2-dimensional neighbor system, where the black dots represents integral vectors in the neighbor system

Matroids. The concept of matroid is introduced by Whitney [21]. One of the important results on matroids, from the viewpoint of combinatorial optimization, is the validity of a greedy algorithm for linear optimization (see, e.g., [11]).
Integral Polymatroids. The concept of polymatroid is introduced by Edmonds [10] as a generalization of matroids. A polymatroid is a polyhedron defined by a monotone submodular function, and a greedy algorithm for matroids can be naturally extends to polymatroids. The minimization of separable-convex function can be also done in a greedy way, and efficient algorithms have been proposed (see, e.g., [13, 15]).
Delta-Matroids. The concept of delta-matroid (or pseudomatroid) is introduced by Bouchet [5] and Chandrasekaran and Kabadi [7]. A delta-matroid can be seen as a family of subsets of a ground set with a nice combinatorial structure, and generalizes the concept of matroid. A more general greedy algorithm works for the linear optimization on a delta-matroid.
Integral Bisubmodular Polyhedron. The concept of bisubmodular polyhedron (or polypseudomatroid), introduced by Chandrasekaran and Kabadi [7] (see also [6, 12]), is a common generalization of polymatroid and delta-matroid. For the following discussion, we give a precise definition. We denote $3^E = \{(X, Y) \mid X, Y \subseteq E,\ X \cap Y = \emptyset\}$. For any $x \in \mathbf{R}^E$ and $(X, Y) \in 3^E$, we define $x(X, Y) = \sum_{i \in X} x(i) - \sum_{i \in Y} x(i)$. A function $\rho : 3^E \to \mathbf{R} \cup \{+\infty\}$ is said to be *bisubmodular* if it satisfies the bisubmodular inequality:

$$\begin{aligned}\rho(X_1, Y_1) + \rho(X_2, Y_2) \geq\ & \rho((X_1 \cup X_2) \setminus (Y_1 \cup Y_2), (Y_1 \cup Y_2) \setminus (X_1 \cup X_2)) \\ & + \rho(X_1 \cap X_2, Y_1 \cap Y_2) \qquad (\forall (X_1, Y_1), (X_2, Y_2) \in 3^E).\end{aligned}$$

For a function $\rho : 3^E \to \mathbf{R} \cup \{+\infty\}$ with $\rho(\emptyset, \emptyset) = 0$, we define a polyhedron $\mathrm{P}_*(\rho) \subseteq \mathbf{R}^E$ by $\mathrm{P}_*(\rho) = \{x \in \mathbf{R}^E \mid x(X, Y) \leq \rho(X, Y)\ ((X, Y) \in 3^E)\}$, which is called a *bisubmodular polyhedron* if ρ is bisubmodular. Bisubmodular polyhedra constitute an important class of polyhedra on which a simple greedy algorithm works for linear optimization. In addition, separable convex function minimization can be done in a greedy manner [2].
Jump Systems. The concept of jump system is introduced by Bouchet and Cunningham [6], which is a common generalization of delta-matroid and the set

of integral vectors in an integral bisubmodular polyhedron. Interesting examples of jump systems can be found in the set of degree sequences of the subgraphs of undirected and directed graphs; for example, matchings and b-matchings in undirected graphs [6, 8, 16] and even-factors in directed graphs [17]. Validity of certain greedy algorithms is shown in [6] for the linear optimization and in [3] for separable-convex function minimization. Moreover, a polynomial-time algorithm for separable-convex function minimization is given in [20].

We give a precise definition of jump systems. For $i \in E$, the characteristic vector $\chi_i \in \{0,1\}^E$ is the vector such that $\chi_i(i) = 1$ and $\chi_i(e) = 0$ for $e \in E \setminus \{i\}$. Denote by U the set of vectors $+\chi_e, -\chi_e$ $(e \in E)$. For vectors $x, y \in \mathbf{Z}^E$, define $\mathrm{inc}(x,y) = \{p \in U \mid x+p \text{ is between } x \text{ and } y\}$. A set $\mathcal{J} \subseteq \mathbf{Z}^E$ is a *jump system* if it satisfies the following axiom:

(J) for every $x, y \in \mathcal{J}$ and every $p \in \mathrm{inc}(x,y)$, if $x + p \notin \mathcal{J}$ then there exists $q \in \mathrm{inc}(x+p, y)$ such that $x + p + q \in \mathcal{J}$.

Note that a jump system is equivalent to an N_2-neighbor system [14].

We give two additional examples of neighbor systems which are not jump systems. The neighbor system in Fig. 1 is also an example of a neighbor system which is not a jump system.

Example 1.1 (Expansion of jump systems). For a jump system $\mathcal{J} \subseteq \mathbf{Z}^E$ and a positive integer k, the set $\{kx \in \mathbf{Z}^E \mid x \in \mathcal{J}\}$ is an N_{2k}-neighbor system [14].

Example 1.2 (Rectilinear grid). Let $u \in \mathbf{Z}^E_+$ be a nonnegative vector, and for $e \in E$, let $\pi_e : [0, u(e)] \to \mathbf{Z}$ be a strictly increasing function. Then, the set of $(\pi_e(x(e)) \mid e \in E)$ for vectors $x \in \mathbf{Z}^E_+$ with $x \leq u$ is an all-neighbor system.

These examples, in particular, show that a neighbor system may have a "hole," as in the case of jump system, and it can be arbitrarily large.

Neighbor systems provide a systematic and simple way to characterize matroids and its generalizations for which greedy algorithms work for linear optimization. Indeed, it is shown that linear optimization on a neighbor system can be solved by a greedy algorithm, and that the greedy algorithm runs in polynomial time for N_k-neighbor systems for every fixed k [14].

The main aim of this paper is to reveal the relationship of neighbor systems with jump systems and with bisubmodular polyhedra. We firstly prove that for every neighbor system $\mathcal{F} \subseteq \mathbf{Z}^E$, there exists a jump system $\mathcal{J} \subseteq \mathbf{Z}^E$ which has the same neighborhood structure as the neighbor system $\mathcal{F}$ (see Th. 3.1). This means that the concept of neighbor system is essentially equivalent to that of jump system, although the class of neighbor systems properly contains that of jump systems. Our result implies that every property of jump systems can be restated in terms of neighbor systems by using the equivalence. Indeed, we show in Section 5 that several useful properties of jump systems naturally extend to neighbor systems.

We then discuss the relationship between neighbor systems and bisubmodular polyhedra. It is known that the convex closure of a jump system, which is a special case of neighbor systems, is an integral bisubmodular polyhedron [6]. We show that the convex closure of a neighbor system is also an integral bisubmodular polyhedron (see Th. 4.1). In addition, we give a characterization of neighbor systems using bisubmodular polyhedra, stating that a set of integral vectors is a neighbor system if and only if the convex closure of its restriction with an interval is always an integral bisubmodular polyhedron (see Th. 4.2). This result implies, in particular, that a simple greedy algorithm for the linear optimization on a bisubmodular polyhedron described below can be also used for neighbor systems (see [9],[12, §3.5 (b)] for the greedy algorithm of this type).

Greedy algorithm for the minimization of a linear function
Step 0: Let x_0 be any vector in $\mathcal{F}$ and put $x := x_0$. Order the elements in $E = \{e_1, e_2, \ldots, e_n\}$ and compute an integer k so that $|w(e_1)| \geq \cdots \geq |w(e_k)| > |w(e_{k+1})| = \cdots = |w(e_n)| = 0$.
Step 1: For $i = 1, 2, \ldots, k$, do the following: if $w(e_i) \geq 0$ (resp., $w(e_i) < 0$), then fix the components $x(e_1), x(e_2), \ldots, x(e_{i-1})$ and decrease (resp., increase) $x(e_i)$ as much as possible under the condition $x \in \mathcal{F}$.

As an application of the results shown in this paper, we consider the separable convex optimization problem on neighbor systems and show that the problem can be solved efficiently. Given a family of univariate convex functions $f_e : \mathbf{Z} \to \mathbf{R}$ ($e \in E$) and a finite neighbor system $\mathcal{F} \subseteq \mathbf{Z}^E$, we consider the following problem:

$$\textbf{(SC)}\ \text{Minimize } f(x) \equiv \textstyle\sum_{e \in E} f_e(x(e)) \text{ subject to } x \in \mathcal{F}.$$

For a special case where $\mathcal{F}$ is a jump system, it is shown that the problem (SC) can be solved in pseudo-polynomial time by a greedy-type algorithm [3], and in weakly-polynomial time by an algorithm called the domain reduction algorithm [20]. We extend these algorithms for jump systems to neighbor systems.

To do this, we show that the problem (SC) on a neighbor system can be reduced to the problem (SC′) of minimizing a separable convex function on a jump system by using the relationship between neighbor systems and jump systems shown in this paper. Note that this reduction does not yield efficient algorithms for neighbor systems since it requires exponential time. Instead, we extend the properties used in the algorithms for jump systems to neighbor systems, which enables us to develop efficient algorithms for neighbor systems.

The organization of this paper is as follows. Section 2 is devoted to preliminaries on the fundamental concepts discussed in this paper. We discuss the relationship of neighbor systems with jump systems and with bisubmodular polyhedra in Sections 3 and 4, respectively. In Section 5, we propose efficient algorithms for (SC). Due to the page limit, most of the proofs are given in Appendix.

2 Preliminaries

We denote by $\mathbf{Z}, \mathbf{Z}_+, \mathbf{Z}_{++}$ the sets of integers, nonnegative integers, and positive integers, respectively. We denote by $\mathbf{R}$ the set of real numbers. For vectors $\ell \in$

$(\mathbf{Z} \cup \{-\infty\})^E$ and $u \in (\mathbf{Z} \cup \{+\infty\})^E$ with $\ell \leq u$, we define the integer interval $[\ell, u]$ as the set of integral vectors $x \in \mathbf{Z}^E$ with $\ell(e) \leq x(e) \leq u(e)$ $(\forall e \in E)$.

We review the original definition of neighbor systems in [14] using the concept of neighbor function. A *neighbor function*, denoted by N, is a function that takes as input any set $\mathcal{F} \subseteq \mathbf{Z}^E$ with any $x \in \mathcal{F}$ and outputs a subset of the neighbors of x in $\mathcal{F}$, denoted by $N(\mathcal{F}, x)$. In particular, $N^a(\mathcal{F}, x)$ (resp., $N_k(\mathcal{F}, x)$) denotes the set of all neighbors of x in $\mathcal{F}$ (resp., the set of all neighbors y of x in $\mathcal{F}$ with $||y - x||_1 \leq k$). For vectors $x, y, z \in \mathbf{Z}^E$, z is said to be *between x and y* if $\min\{x(e), y(e)\} \leq z(e) \leq \max\{x(e), y(e)\}$ $(\forall e \in E)$. Given a set $\mathcal{F} \subseteq \mathbf{Z}^E$ and a neighbor function N, we say that $\mathcal{F}$ is an *N-neighbor system* if the following condition holds:

> **(NNS)** for every $x, y \in \mathcal{F}$ and every $i \in E$ with $x(i) \neq y(i)$, there exists $z \in N(\mathcal{F}, x)$ such that z is between x and y and $z(i) \neq x(i)$.

An N-neighbor system is an all-neighbor system if $N = N^a$, and an N_k-neighbor system if $N = N_k$.

In the following discussion, we use an equivalent axiom of neighbor systems given below. Then, (NNS) can be rewritten as follows:

> **(NNS′)** for every $x, y \in \mathcal{F}$ and every $p \in \mathrm{inc}(x, y)$, there exist $q \in \mathrm{inc}(x, y) \cup \{\mathbf{0}\} \setminus \{p\}$ and $\alpha \in \mathbf{Z}_{++}$ such that $x' \equiv x + \alpha(p + q) \in N(\mathcal{F}, x)$ and x' is between x and y.

We note that the axiom (NNS′) is similar to the axiom (J) for jump systems.

The class of neighbor systems is closed under the following operations.

Proposition 2.1 (cf. [14]). *Let $\mathcal{F} \subseteq \mathbf{Z}^n$ be an N-neighbor system.*
(i) *For a positive integer $m > 0$, define a set $\mathcal{F}' = \{mx \mid x \in \mathcal{F}\}$ and a neighbor function N' by $N'(\mathcal{F}', mx) = \{my \mid y \in N(\mathcal{F}, x)\}$. Then, $\mathcal{F}'$ is an N'-neighbor system.*
(ii) *For a vector $s \in \{+1, -1\}^E$, we define a set $\mathcal{F}_s = \{(s(e)x(e) \mid e \in E) \mid x \in \mathcal{F}\}$ and a neighbor function N_s by $N_s(\mathcal{F}_s, y) = \{(s(e)x'(e) \mid e \in E) \mid x' \in N(\mathcal{F}, x)\}$ for $y = (s(e)x(e) \mid e \in E) \in \mathcal{F}_s$. Then, $\mathcal{F}_s$ is an N_s-neighbor system.*
(iii) *For vectors $\ell, u \in \mathbf{Z}^E$ with $\ell \leq u$ and $\mathcal{F} \cap [\ell, u] \neq \emptyset$, the set $\mathcal{F} \cap [\ell, u]$ is an N-neighbor system.*
(iv) *For a vector $a \in \mathbf{Z}^E$, the set $\{x + a \mid x \in \mathcal{F}\}$ is an N'-neighbor system, where $N'(\mathcal{F} + a, x) = \{y + a \mid y \in N(\mathcal{F}, x)\}$.*

We introduce a concept of *proper neighbor* for neighbor systems. For $p \in U$, we define $\mathrm{e}(p)$ to be the element $e \in E$ satisfying $p \in \{+\chi_e, -\chi_e\}$. For a neighbor system $\mathcal{F}$ and vectors $x, y \in \mathcal{F}$, we say that y is a *proper neighbor of x in $\mathcal{F}$* if y is a neighbor of x satisfying either of the conditions (i) or (ii), where

> (i) there exist some $\alpha \in \mathbf{Z}_{++}$ and $p \in U$ such that $y - x = \alpha p$,
> (ii) there exist some $\alpha \in \mathbf{Z}_{++}$ and $p, q \in U$ with $\mathrm{e}(p) \neq \mathrm{e}(q)$ such that $y - x = \alpha(p + q)$ and $x + \alpha' p \notin \mathcal{F}$ for all α' with $0 < \alpha' \leq \alpha$.

To illustrate the concept of proper neighbor, consider the neighbor system in Fig. 1. The vector $(6,2)$ is a proper neighbor of $(4,0)$ since $(5,0)$ and $(6,0)$ are not in $\mathcal{F}$. The vector $(6,5)$ is a neighbor of $(3,2)$, but not a proper neighbor of $(3,2)$ since $(4,2),(3,5) \in \mathcal{F}$.

For jump system, which is a special case of neighbor system, the definition of proper neighbor can be simplified as follows; for a jump system $\mathcal{J}$ and vectors $x,y \in \mathcal{J}$, the vector y is said to be a *proper neighbor of x in $\mathcal{J}$* if it satisfies either of the conditions (i) and (ii), where

(i) $y - x = p$ or $y - x = 2p$ for some $p \in U$,
(ii) $y - x = p + q$ for some $p, q \in U$ such that $q \neq -p$ and $x + p \notin \mathcal{J}$.

3 Relationship between Neighbor Systems and Jump Systems

We discuss the relationship between neighbor systems and jump systems. It is shown that for every neighbor system, there exists a jump system which has the same neighborhood structure as the original neighbor system.

Theorem 3.1. *Let $\mathcal{F} \subseteq \mathbf{Z}^E$ be an all-neighbor system. Then, there exist a jump system $\mathcal{J} \subseteq \mathbf{Z}^E$ and a bijective function $\pi : \mathcal{J} \to \mathcal{F}$ satisfying the following property:*

for every $x, y \in \mathcal{F}$, the vector x is a proper neighbor of y in $\mathcal{F}$ if and only if $\pi^{-1}(x)$ is a proper neighbor of $\pi^{-1}(y)$ in $\mathcal{J}$, where $\pi^{-1} : \mathcal{F} \to \mathcal{J}$ is the inverse function of π.

Proof. Let $\mathcal{F} \subseteq \mathbf{Z}^E$ be an all-neighbor system. By Proposition 2.1 (iv), we may assume, without loss of generality, that $\mathcal{F}$ contains the zero vector $\mathbf{0}$. For $e \in E$, we define a set $\mathcal{F}_e \subseteq \mathbf{Z}$ by

$$\mathcal{F}_e = \{\alpha \mid \alpha \in \mathbf{Z},\ \exists x \in \mathcal{F} \text{ s.t. } x(e) = \alpha\}.$$

Define the numbers $u(e) \in \mathbf{Z} \cup \{+\infty\}$ and $\ell(e) \in \mathbf{Z} \cup \{-\infty\}$ by

$$\begin{aligned} u(e) &= \text{the number of positive integers in } \mathcal{F}_e, \\ l(e) &= -(\text{the number of negative integers in } \mathcal{F}_e). \end{aligned}$$

We also define a function $\pi_e : [\ell(e), u(e)] \to \mathbf{Z}$ by $\pi_e(0) = 0$ and

$$\begin{aligned} \pi_e(k) &= \text{ the } k\text{-th smallest positive integer in } \mathcal{F}_e \qquad (\text{if } 0 < k \leq u(e)), \\ \pi_e(-k) &= \text{ the } k\text{-th largest negative integer in } \mathcal{F}_e \qquad (\text{if } \ell(e) \leq -k < 0). \end{aligned}$$

Then, each π_e is a strictly increasing function in the interval $[\ell(e), u(e)]$. We define a set $\mathcal{J} \subseteq \mathbf{Z}^E$ and a function $\pi : \mathcal{J} \to \mathcal{F}$ by

$$\mathcal{J} = \{z \in \mathbf{Z}^E \mid (\pi_e(z(e)) \mid e \in E) \in \mathcal{F}\}, \qquad \pi(z) = (\pi_e(z(e)) \mid e \in E) \quad (z \in \mathcal{J}).$$

By the definitions of π_e and $\mathcal{J}$, the function π is bijective.

To complete the proof of Theorem 3.1, it suffices to show the following:

Lemma 3.1.
(i) *The set $\mathcal{J}$ is a jump system.*
(ii) *For every $x, y \in \mathcal{F}$, the vector x is a proper neighbor of y in $\mathcal{F}$ if and only if $\pi^{-1}(x)$ is a proper neighbor of $\pi^{-1}(y)$ in $\mathcal{J}$.*

The proof of (i) and (ii) are given in Sections A.1 and A.2, respectively. □

4 Polyhedral Structure of Neighbor Systems

We prove the following theorems concerning the polyhedral structure of the convex closure of neighbor systems. For a set $\mathcal{F} \subseteq \mathbf{Z}^n$, we denote by $\mathrm{conv}(\mathcal{F})$ ($\subseteq \mathbf{R}^n$) the convex closure (closed convex hull) of $\mathcal{F}$.

Theorem 4.1. *For every all-neighbor system $\mathcal{F} \subseteq \mathbf{Z}^E$, its convex closure* $\mathrm{conv}(\mathcal{F})$ *is an integral bisubmodular polyhedron.*

It should be noted that Theorem 4.1 does not follow immediately from Theorem 3.1 and the fact that the convex closure of a jump system is an integral bisubmodular polyhedron [6].

We also provide a characterization of neighbor systems by the property that the convex closure is a bisubmodular polyhedron.

Theorem 4.2. *A nonempty set $\mathcal{F} \subseteq \mathbf{Z}^E$ is an all-neighbor system if and only if for all vectors $\ell, u \in \mathbf{Z}^E$ satisfying $\ell \leq u$ and $\mathcal{F} \cap [\ell, u] \neq \emptyset$, the convex closure* $\mathrm{conv}(\mathcal{F} \cap [\ell, u])$ *is an integral bisubmodular polyhedron.*

Below we give proofs of Theorems 4.1 and 4.2.

Proof of Theorem 4.1. Let $\mathcal{F} \subseteq \mathbf{Z}^E$ be a neighbor system, and $\rho : 3^E \to \mathbf{R} \cup \{+\infty\}$ is a function defined by $\rho(X, Y) = \sup\{x(X) - x(Y) \mid x \in \mathcal{F}\}$ ($(X, Y) \in 3^E$). Note that $\rho(\emptyset, \emptyset) = 0$ and the value $\rho(X, Y)$ is integer if $\rho(X, Y) < +\infty$. To prove Theorem 4.1, it suffices to show that the function ρ is a bisubmodular function satisfying $\mathrm{conv}(\mathcal{F}) = \mathrm{P}_*(\rho)$.

We here consider only the case where $\mathcal{F}$ is a finite set; the case where $\mathcal{F}$ is not necessarily a finite set is given in Section A.6. Then, $\rho(X, Y) < +\infty$ holds for all $(X, Y) \in 3^E$. We give a key property to show Theorem 4.1, where the proof is given in Section A.3.

Lemma 4.1. *For every $(A, B) \in 3^E$ with $A \cup B = E$ and all subsets $V_1, V_2, \ldots, V_k$ ($k \geq 1$) of E with $V_1 \subset V_2 \subset \cdots \subset V_k$, there exists some $x \in \mathcal{F}$ such that $x(V_t \cap A, V_t \cap B) = \rho(V_t \cap A, V_t \cap B)$ for $t = 1, 2, \ldots, k$.*

To show the bisubmodularity of ρ, we use the following characterization.

Lemma 4.2 ([4, Th. 2]). *A function $\rho : 3^E \to \mathbf{R}$ is bisubmodular if and only if ρ satisfies the following conditions:*

$$\begin{aligned}
&\rho(X \cap A, X \cap B) + \rho(Y \cap A, Y \cap B) \\
&\quad \geq \rho((X \cup Y) \cap A, (X \cup Y) \cap B) + \rho((X \cap Y) \cap A, (X \cap Y) \cap B) \\
&\qquad\qquad\qquad\qquad (\forall (A, B) \in 3^E \text{ with } A \cup B = E, \forall X, Y \in 2^E), \quad (1) \\
&\rho(X \cup \{i\}, Y) + \rho(X, Y \cup \{i\}) \geq 2\rho(X, Y) \\
&\qquad\qquad\qquad\qquad (\forall (X, Y) \in 3^E,\ \forall i \in E \setminus (X \cup Y)). \quad (2)
\end{aligned}$$

Note that the condition (1) is equivalent to the submodularity of the function $\rho_{A,B} : 2^N \to \mathbf{R}$ defined by $\rho_{A,B}(X) = \rho(X \cap A, X \cap B)$ $(X \in 2^N)$. By using this characterization, we prove that the function ρ is bisubmodular, where the proof is given in Section A.4.

Lemma 4.3. *The function ρ is bisubmodular.*

To show the equation $\mathrm{conv}(\mathcal{F}) = \mathrm{P}_*(\rho)$, we use the following characterization of extreme points in a bounded bisubmodular polyhedron.

Lemma 4.4 ([12, Cor. 3.59]). *Let $\rho : 3^E \to \mathbf{R}$ be a bisubmodular function. A vector $x \in \mathbf{R}^E$ is an extreme point of $\mathrm{P}_*(\rho)$ if and only if there exist $(A, B) \in 3^E$ with $A \cup B = E$ and subsets $V_0, V_1, \ldots, V_n$ of E with $\emptyset = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = E$ such that $x(V_t \cap A, V_t \cap B) = \rho(V_t \cap A, V_t \cap B)$ for $t = 1, 2, \ldots, n$.*

Lemma 4.5. *It holds that $\mathrm{conv}(\mathcal{F}) = \mathrm{P}_*(\rho)$.*

Proof. By the definition of $\mathrm{P}_*(\rho)$, it is easy to see that $\mathrm{conv}(\mathcal{F}) \subseteq \mathrm{P}_*(\rho)$. To show the reverse inclusion, it suffices to show that every extreme point of $\mathrm{P}_*(\rho)$ is contained in $\mathcal{F}$, which follows from Lemmas 4.1 and 4.4. □

Proof of Theorem 4.2. The "only if" part is immediate from Theorem 4.1 and Proposition 2.1 (iii). In the following, we prove the "if" part. Let $x, y \in \mathcal{F}$ and $i \in E$ with $x(i) \neq y(i)$. Assume, without loss of generality, that $x(i) > y(i)$.

Lemma 4.6. *There exists some $z_0 \in \mathcal{F}$ such that z_0 is between x and y and $z_0(i) > x(i)$.*

The proof of this lemma is given in Section A.5. Let α be the minimum positive number such that $\alpha z_0 + (1 - \alpha)x \in \mathcal{F}$, and put $z = \alpha z_0 + (1 - \alpha)x$. Then, z is a neighbor of x between x and y and satisfies $z(i) > x(i)$. This concludes the proof of Theorem 4.2.

5 Separable Convex Optimization on Neighbor Systems

We consider the problem (SC) of minimizing a separable convex function on a finite neighbor system. We propose a greedy algorithm for the problem (SC) and show that it runs in pseudo-polynomial time. We then show that the problem

(SC) can be solved in weakly polynomial time if $\mathcal{F}$ is an N_k-neighbor system with a fixed k.

We put $n = |E|$, and define the *size* of $\mathcal{F}$ by $\Phi(\mathcal{F}) = \max_{e \in E}[\max_{x \in \mathcal{F}} x(e) - \min_{x \in \mathcal{F}} x(e)]$. It is assumed that we are given a membership oracle for $\mathcal{F}$, which enables us to check whether a given vector is contained in $\mathcal{F}$ or not in constant time. For simplicity, we mainly assume in this section that $\mathcal{F}$ is an N_k-neighbor system for some k; note that a finite all-neighbor system can be seen as an N_k-neighbor system with $k = \Phi(\mathcal{F})$.

5.1 Theorems

We show some useful properties in developing efficient algorithms for (SC). The next theorem shows that the optimality of a vector can be characterized by a local optimality.

Theorem 5.1. *A vector $x \in \mathcal{F}$ is an optimal solution of* (SC) *if and only if $f(x) \leq f(y)$ for every proper neighbor y of x.*

The next property shows that a given nonoptimal vector in $\mathcal{F}$ can be easily separated from an optimal solution.

Theorem 5.2. *Let $x \in \mathcal{F}$ be a vector which is not an optimal solution of* (SC)*. Let $x' \equiv x + \alpha_*(p_* + q_*)$ be a proper neighbor of x in $\mathcal{F}$ such that $p_* \in U$, $q_* \in U \cup \{\mathbf{0}\} \setminus \{+p_*, -p_*\}$, $\alpha_* \in \mathbf{Z}_{++}$, and $f(x') < f(x)$. Suppose that x' minimizes the value $\{f(x + \alpha_* p_*) - f(x)\}/\alpha_*$ among all such vectors. Then, there exists an optimal solution x_* of* (SC) *satisfying*

$$\begin{cases} x_*(i) \leq x(i) - \alpha_- \ (\mathit{if}\ p_* = -\chi_i), \\ x_*(i) \geq x(i) + \alpha_+ \ (\mathit{if}\ p_* = +\chi_i), \end{cases}$$

where $\alpha_- = \min\{x(i) - y(i) \mid y \in \mathcal{F},\ y(i) < x(i)\}$ and $\alpha_+ = \min\{y(i) - x(i) \mid y \in \mathcal{F},\ y(i) > x(i)\}$.

To prove Theorems 5.1 and 5.2, we show that the problem (SC) can be reduced to the problem (SC′) of minimizing a separable convex function on a jump system by using the relationship between neighbor systems and jump systems shown in Section 3.

We define vectors $\ell, u \in \mathbf{Z}^E$, a jump system $\mathcal{J} \subseteq \mathbf{Z}^E$, and a family of strictly increasing functions $\pi_e : [\ell(e), u(e)] \to \mathbf{Z}$ $(e \in E)$ as in Section 3. We define functions $g_e : [\ell(e), u(e)] \to \mathbf{R}$ $(e \in E)$ by

$$g_e(\alpha) = f_e(\pi_e(\alpha)) \qquad (\alpha \in [\ell(e), u(e)]).$$

Note that g_e is a convex function since f_e is a convex function. Then, the problem (SC) for a neighbor system $\mathcal{F}$ can be reduced to the following problem:

(SC′) Minimize $g(x) \equiv \sum_{e \in E} g_e(x(e))$ subject to $x \in \mathcal{J}$, which is the minimization of a separable convex function g on a jump system $\mathcal{J}$.

For the problem (SC′), the following properties are known.

Theorem 5.3 (cf. [3, Cor. 4.2]). *A vector $x \in \mathcal{J}$ is an optimal solution of* (SC′) *if and only if $g(x) \leq g(y)$ for all proper neighbors y of x in $\mathcal{J}$.*

Theorem 5.4 (cf. [20, Th. 4.2]). *Let $x \in \mathcal{J}$ be a vector that is not an optimal solution of* (SC′)*. Let $x' \equiv x + p_* + q_*$ be a proper neighbor of x in $\mathcal{J}$ such that $p_* \in U$, $q \in U \cup \{\mathbf{0}\}$, and $g(x') < g(x)$, and suppose that x' minimizes the value $g(x + p_*)$ among all such vectors. Then, there exists an optimal solution $x_* \in \mathcal{J}$ of* (SC′) *satisfying $x_*(i) \leq x(i) - 1$ if $p_* = -\chi_i$ and $x_*(i) \geq x(i) + 1$ if $p_* = +\chi_i$.*

Then, Theorems 5.1 and 5.2 are just the restatement of Theorems 5.3 and 5.4 by using Theorem 3.1 and the equivalence between (SC) and (SC′).

We then show that the check of local optimality in the sense of Theorem 5.1 and the computation of a proper neighbor x' in Theorem 5.2 can be done efficiently. The proof is given in Section A.7.

Theorem 5.5. *Let $\mathcal{F} \subseteq \mathbf{Z}^E$ be an N_k-neighbor system. For $x \in \mathcal{F}$, all proper neighbors of x can be computed in $\mathrm{O}(n^2 k)$ time.*

5.2 Pseudopolynomial-Time Algorithm

Based on Theorems 5.1 and 5.2, we propose a greedy algorithm for solving the problem (SC). The greedy algorithm maintains an interval $[a, b]$, where $a, b \in \mathbf{Z}^E$ containing an optimal solution of (SC). Note that $\mathcal{F} \cap [a, b]$ is a neighbor system by Proposition 2.1 (iii). The vectors a and b are updated by using Theorem 5.2 so that the value $||b - a||_1$ reduces in each iteration, and finally an optimal solution is found. Recall that $\mathcal{F}$ is assumed to be a finite N_k-neighbor system. We assume that an initial vector $x_0 \in \mathcal{F}$ is given.

Algorithm Greedy
Step 0: Let $x := x_0 \in \mathcal{F}$. Set $a(e) := a_{\mathcal{F}}(e)$ and $b(e) := b_{\mathcal{F}}(e)$, where

$$a_{\mathcal{F}}(e) := \min\{x(e) \mid x \in \mathcal{F}\}, \quad b_{\mathcal{F}}(e) := \max\{x(e) \mid x \in \mathcal{F}\} \qquad (e \in E). \tag{3}$$

Step 1: If $f(x) \leq f(y)$ for all proper neighbors y of x in $\mathcal{F} \cap [a, b]$, then stop (x is optimal).
Step 2: Let $x' \equiv x + \alpha_*(p_* + q_*)$ be a proper neighbor of x in $\mathcal{F} \cap [a, b]$ such that $p_* \in U$, $q_* \in U \cup \{\mathbf{0}\} \setminus \{+p_*, -p_*\}$, $\alpha_* \in \mathbf{Z}_{++}$, and $f(x') < f(x)$, and suppose that x' minimizes the value $\{f(x + \alpha_* p_*) - f(x)\}/\alpha_*$ among all such vectors.
Step 3: Modify a or b as follows:

$$\begin{cases} b(i) := x(i) - \alpha_- \ (\text{if } p_* = -\chi_i), \\ a(i) := x(i) + \alpha_+ \ (\text{if } p_* = +\chi_i), \end{cases} \tag{4}$$

where α_-, α_+ are defined by

$$\begin{aligned} \alpha_- &= \min\{x(i) - y(i) \mid y \in \mathcal{F} \cap [a, b],\ y(i) < x(i)\}, \\ \alpha_+ &= \min\{y(i) - x(i) \mid y \in \mathcal{F} \cap [a, b],\ y(i) > x(i)\}. \end{aligned} \tag{5}$$

Set $x := x'$. Go to Step 1. □

We show the validity of the algorithm. By Theorem 5.1, the output x of the algorithm is a minimizer of the function f in the set $\mathcal{F} \cap [a, b]$. We see from Theorem 5.2 that the set $\mathcal{F} \cap [a, b]$ always contains an optimal solution of (SC). Hence, the output x of the algorithm is an optimal solution of (SC).

Time complexity analysis is given in Section A.8.

Theorem 5.6. *The algorithm* GREEDY *finds an optimal solution of the problem* (SC). *The running time is* $\mathrm{O}(n^3\,\Phi(\mathcal{F})^2)$ *if* $\mathcal{F}$ *is an all-neighbor system, and* $\mathrm{O}(n^3\,k\,\Phi(\mathcal{F}))$ *if* $\mathcal{F}$ *is an* N_k*-neighbor system and the value* k *is given.*

5.3 Polynomial-Time Algorithm

We propose a faster algorithm for (SC) based on the domain reduction approach. The domain reduction approach is used in [19, 20] to develop polynomial-time algorithms for various discrete convex function minimization problems. We show that the proposed algorithm runs in weakly polynomial time if $\mathcal{F}$ is an N_k-neighbor system with a fixed k and the value k is known a priori.

Given an N_k-neighbor system $\mathcal{F} \subseteq \mathbf{Z}^E$, we define a set $\mathcal{F}^\bullet \subseteq \mathbf{Z}^E$ by $\mathcal{F}^\bullet = \mathcal{F} \cap [a^\bullet_{\mathcal{F}}, b^\bullet_{\mathcal{F}}]$, where $a_{\mathcal{F}}, b_{\mathcal{F}} \in \mathbf{Z}^E$ are defined by (3) and

$$a^\bullet_{\mathcal{F}}(e) = a_{\mathcal{F}}(e) + \left\lfloor \frac{b_{\mathcal{F}}(e)-a_{\mathcal{F}}(e)}{nk} \right\rfloor, \quad b^\bullet_{\mathcal{F}}(e) = b_{\mathcal{F}}(e) - \left\lfloor \frac{b_{\mathcal{F}}(e)-a_{\mathcal{F}}(e)}{nk} \right\rfloor \quad (e \in E).$$

The following properties of the set $\mathcal{F}^\bullet$ are proved in Section A.9.

Theorem 5.7. *Let* $\mathcal{F} \subseteq \mathbf{Z}^E$ *be an* N_k*-neighbor system.*
(i) *The set* $\mathcal{F}^\bullet$ *is nonempty and hence an* N_k*-neighbor system.*
(ii) *A vector in* $\mathcal{F}^\bullet$ *can be found in* $\mathrm{O}(n^3 k \log \Phi(\mathcal{F}))$ *time, provided a vector in* $\mathcal{F}$ *is given.*

The algorithm is as follows. Assume that an initial vector $x_0 \in \mathcal{F}$ is given.

Algorithm DOMAIN_REDUCTION
Step 0: Set $a := a_{\mathcal{F}}$ and $b := b_{\mathcal{F}}$.
Step 1: Find a vector $x \in (\mathcal{F} \cap [a, b])^\bullet$.
Step 2: If $f(x) \le f(y)$ for all proper neighbors y of x in $\mathcal{F} \cap [a, b]$, then stop.
Step 3: Let $x' \equiv x + \alpha_*(p_* + q_*)$ be a proper neighbor of x in $\mathcal{F} \cap [a, b]$ satisfying the same condition as in Step 2 of GREEDY.
Step 4: Modify a or b by (4). Go to Step 1. □

The validity of this algorithm can be shown in a similar way as the algorithm GREEDY. The analysis of the time complexity is given in Section A.10, where the following property is the key to obtain a polynomial bound.

Lemma 5.1. *Let* p_* *be the vector chosen in Step 3 of the* m*-th iteration, and* $i = \mathrm{e}(p_*) \in E$. *Then, we have* $b_{m+1}(i) - a_{m+1}(i) < (1 - 1/nk)(b_m(i) - a_m(i))$.

Theorem 5.8. *The algorithm* DOMAIN_REDUCTION *finds an optimal solution of the problem* (SC) *in* $\mathrm{O}(n^5 k^2 (\log \Phi(\mathcal{F}))^2)$ *time if* $\mathcal{F}$ *is an* N_k*-neighbor system.*

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A Appendix: Proofs

A.1 Proof of Lemma 3.1 (i)

To prove Lemma 3.1 (i), we use the following lemmas.

Lemma A.1. *Let $z \in \mathcal{F}$ and suppose that $z + \alpha p + \beta q \in \mathcal{F}$ holds for some $p, q \in U$ with $\mathrm{e}(p) \neq \mathrm{e}(q)$ and $\alpha, \beta \in \mathbf{Z}_+$. Then, there exists some $\alpha' \in \mathbf{Z}$ such that $0 \leq \alpha' \leq \alpha,\ \alpha - \alpha' \leq \beta$, and $z + \alpha' p \in \mathcal{F}$.*

Proof. Consider a vector $z' \in \mathcal{F}$ of the form $z' = z + (\alpha - \varepsilon)p + (\beta - \delta)q$, where ε and δ are nonnegative integers satisfying $\varepsilon \leq \alpha$, $\delta \leq \beta$, and $\varepsilon \leq \delta$. Note that such a vector exists when $\varepsilon = \delta = 0$. Suppose that the vector z' maximizes the value δ among all such vectors.

Suppose, to the contrary, that $\delta < \beta$. Since $-q \in \mathrm{inc}(z', z)$, the property (NNS′) implies that there exists some integers ε', δ' such that $0 \leq \varepsilon' \leq \alpha - \varepsilon$, $0 < \delta' \leq \beta - \delta$, $\varepsilon' \in \{0, \delta'\}$, and

$$z' - \varepsilon' p - \delta' q = z + (\alpha - \varepsilon - \varepsilon')p + (\beta - \delta - \delta')q \in \mathcal{F},$$

which is a contradiction to the choice of z'. Hence, we have $\delta = \beta$. Then, it holds that $z' = z + (\alpha - \varepsilon)p \in \mathcal{F}$ and $0 \leq \varepsilon \leq \min\{\alpha, \beta\}$, i.e., the statement of the lemma holds.

Lemma A.2. *Let $z \in \mathcal{F}$, and $p_1, p_2, p_3 \in U$ be vectors such that the elements $\mathrm{e}(p_1), \mathrm{e}(p_2), \mathrm{e}(p_3)$ are distinct. Suppose that $z + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 \in \mathcal{F}$ holds for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{Z}_+$. Then, there exist some integers α_1' and α_2' such that $0 \leq \alpha_1' \leq \alpha_1$, $0 \leq \alpha_2' \leq \alpha_2$, $(\alpha_1 - \alpha_1') + (\alpha_2 - \alpha_2') \leq \alpha_3$, and $z + \alpha_1' p_1 + \alpha_2' p_2 \in \mathcal{F}$.*

Proof. The proof is similar to that for Lemma A.1 and therefore omitted.

We now show that $\mathcal{J}$ is a jump system. Let $\tilde{x}, \tilde{y} \in \mathcal{J}$, and $p \in \mathrm{inc}(\tilde{x}, \tilde{y})$, and suppose that $x + p \notin \mathcal{J}$. By Proposition 2.1 (ii), we may assume that $p = +\chi_i$ for some $i \in E$. We will show that

$$\exists q \in \mathrm{inc}(\tilde{x} + \chi_i, \tilde{y}) \text{ such that } \tilde{x} + \chi_i + q \in \mathcal{J}. \tag{6}$$

Define $x, y \in \mathcal{F}$ by $x = \pi(\tilde{x})$ and $y = \pi(\tilde{y})$. Then, we have $+\chi_i \in \mathrm{inc}(x, y)$ since $+\chi_i \in \mathrm{inc}(\tilde{x}, \tilde{y})$ and each π_e is a strictly increasing function.

We firstly consider the case where there exists some $\alpha_* \in \mathbf{Z}_{++}$ such that $x + \alpha_* \chi_i \in \mathcal{F}$ and $0 < \alpha_* \leq y(i) - x(i)$. We note that $\tilde{x}(i) < \tilde{y}(i) \leq u(i)$ since $+\chi_i \in \mathrm{inc}(\tilde{x}, \tilde{y})$.

Lemma A.3. *Suppose that there exists some $\alpha_* \in \mathbf{Z}_{++}$ such that $x + \alpha_* p \in \mathcal{F}$ and $0 < \alpha_* \leq y(i) - x(i)$. Then, we have either*

(a) $x + (\pi_i(\tilde{x}(i) + 1) - \pi_i(\tilde{x}(i)))\chi_i \in \mathcal{F}$, *or*
(b) $u(i) \geq \tilde{y}(i) \geq \tilde{x}(i) + 2$ *and* $x + (\pi_i(\tilde{x}(i) + 2) - \pi_i(\tilde{x}(i)))\chi_i \in \mathcal{F}$ *(or both).*

Proof. Put $\alpha_1 = \pi_i(\tilde{x}(i)+1) - \pi_i(\tilde{x}(i))$. Suppose firstly that $y(i) - x(i) \leq \alpha_1$ holds. By the definition of the function π_i, we have $\{x' \in \mathcal{F} \mid x(i) < x'(i) < x(i) + \alpha_1\} = \emptyset$. Hence, we have $y(i) - x(i) = \alpha_1$ holds. Since $x + \alpha_* p$ is between x and y, we have $\alpha_* = \alpha_1$, i.e., $x + \alpha_1 \chi_i \in \mathcal{F}$ holds.

We then suppose that $y(i) - x(i) > \alpha_1$ holds. Then, it holds that $u(i) \geq \tilde{y}(i) \geq \tilde{x}(i) + 2$ and $y(i) - x(i) \geq \alpha_2$, where $\alpha_2 = \pi_i(\tilde{x}(i)+2) - \pi_i(\tilde{x}(i))$. In the following, we assume

$$x + \alpha_1 \chi_i \notin \mathcal{F}, \qquad x + \alpha_2 \chi_i \notin \mathcal{F}, \tag{7}$$

and derive a contradiction. We may assume that α_* is the minimum positive integer with $x + \alpha_* \chi_i \in \mathcal{F}$, implying that

$$x + \alpha' \chi_i \notin \mathcal{F} \qquad \text{for all } \alpha' \text{ with } 0 < \alpha' < \alpha_*. \tag{8}$$

Claim 1: There exists some $q \in U \setminus \{+\chi_i\}$ such that $x + \alpha_1(\chi_i + q) \in \mathcal{F}$.
[Proof of Claim 1] Let $z \in \mathcal{F}$ be a vector with $z(i) = \pi_i(\tilde{x}(i)+1) = x(i) + \alpha_1$. Since $+\chi_i \in \mathrm{inc}(x, z)$, the property (NNS′) implies that there exist some $q \in \mathrm{inc}(x,z) \cup \{\mathbf{0}\} \setminus \{+\chi_i\}$ and $\gamma \in \mathbf{Z}_{++}$ such that $x + \gamma(\chi_i + q) \in \mathcal{F}$ and $x + \gamma(\chi_i + q)$ is between x and z. It follows that $0 < \gamma \leq z(i) - x(i) = \alpha_1$, implying $\gamma = \alpha_1$ since $\{x' \in \mathcal{F} \mid x(i) < x'(i) < x(i) + \alpha_1\} = \emptyset$. By (7), we have $q \neq \mathbf{0}$ [End of Claim 1]

Claim 2: $\alpha_2 < \alpha_* \leq 2\alpha_1$.
[Proof of Claim 2] By (7), we have $\alpha_* > \alpha_2$. Suppose, to the contrary, that $\alpha_* > 2\alpha_1$. Consider the vectors $x + \alpha_* \chi_i \in \mathcal{F}$ and $x + \alpha_1(\chi_i + q) \in \mathcal{F}$. By Lemma A.1, there exists some $\eta \in \mathbf{Z}$ such that $\alpha_1 \leq \eta \leq \alpha_*$, $\eta - \alpha_1 \leq \alpha_1$, and $x + \eta \chi_i \in \mathcal{F}$. This, however, is a contradiction to (8) since $\eta \leq 2\alpha_1 < \alpha_*$. [End of Claim 2]

Let $z \in \mathcal{F}$ be a vector with $z(i) = \pi_i(\tilde{x}(i)+2) = x(i) + \alpha_2$. We have $-\chi_i \in \mathrm{inc}(x + \alpha_* \chi_i, z)$ by Claim 2. Hence, there exists some $s \in \mathrm{inc}(x + \alpha_* \chi_i, z) \cup \{\mathbf{0}\} \setminus \{-\chi_i\}$ and $\mu \in \mathbf{Z}_{++}$ such that $x + (\alpha_* - \mu)\chi_i + \mu s \in \mathcal{F}$ and $x + (\alpha_* - \mu)\chi_i + \mu s$ is between $x + \alpha_* \chi_i$ and v. It follows that $\alpha_2 \leq \alpha_* - \mu < \alpha_*$, implying $\mu \leq \alpha_* - \alpha_2$. By Lemma A.1 applied to x and $x + (\alpha_* - \mu)\chi_i + \mu s$, we have some $\gamma \in \mathbf{Z}$ such that $\max\{0, \alpha_* - 2\mu\} \leq \gamma \leq \alpha_* - \mu$ and $x + \gamma \chi_i \in \mathcal{F}$. Then, we have $\gamma < \alpha_*$ and

$$\gamma \geq \alpha_* - 2\mu \geq \alpha_* - 2(\alpha_* - \alpha_2) = -\alpha_* + 2\alpha_2 > -\alpha_* + 2\alpha_1 \geq 0,$$

where the last inequality is by Claim 2. This, however, is a contradiction to (8) since $x + \gamma \chi_i \in \mathcal{F}$.

Lemma A.3 implies that we have either (a) $\tilde{x} + \chi_i \in \mathcal{J}$ or (b) $u(i) \geq \tilde{y}(i) \geq \tilde{x}(i) + 2$ and $\tilde{x} + 2\chi_i \in \mathcal{J}$ (or both). Since $\tilde{x} + \chi_i \notin \mathcal{J}$ by assumption, we have (6) with $q = +\chi_i$.

We then assume that

$$x + \alpha \chi_i \notin \mathcal{F} \qquad (0 < \forall \alpha \leq y(i) - x(i)). \tag{9}$$

Since $+\chi_i \in \mathrm{inc}(x, y)$, the property (NNS$'$) implies that there exist $q \in \mathrm{inc}(x, y) \cup \{\mathbf{0}\} \setminus \{+\chi_i\}$ and $\beta \in \mathbf{Z}_{++}$ such that $x+\beta(\chi_i+q)$ is a neighbor of x and between x and y. By Proposition 2.1 (ii), we may assume that $q = +\chi_j$ for some $j \in E \setminus \{i\}$. Since $x + \beta(\chi_i + \chi_j)$ is a neighbor of x, we have

$$x + \alpha'(\chi_i + \chi_j) \notin \mathcal{F} \qquad (0 < \forall\alpha' < \beta). \tag{10}$$

Lemma A.4. *For every $\beta', \beta'' \in \mathbf{Z}$ such that $0 \le \beta' \le \beta$ and $0 \le \beta'' \le \beta$, if $x + \beta'\chi_i + \beta''\chi_j \in \mathcal{F}$ then we have $(\beta', \beta'') \in \{(0, 0), (0, \beta), (\beta, \beta)\}$.*

Proof. Let $\beta', \beta'' \in \mathbf{Z}$ be such that $0 \le \beta' \le \beta$, $0 \le \beta'' \le \beta$, and $x+\beta'\chi_i+\beta''\chi_j \in \mathcal{F}$. It suffices to show that $\beta', \beta'' \in \{0, \beta\}$ since $x + \beta\chi_i \notin \mathcal{F}$ by (9).

Suppose, to the contrary, that $0 < \beta' < \beta$ holds. Since $+\chi_i \in \mathrm{inc}(x, x + \beta'\chi_i + \beta''\chi_j)$, the property (NNS$'$) implies that there exists some $\eta \in \mathbf{Z}$ with $0 < \eta \le \beta' < \beta$ such that either $x + \eta(\chi_i + \chi_j) \in \mathcal{F}$ or $x + \eta\chi_i \in \mathcal{F}$ (or both). This, however, contradicts (10) or (9).

We then suppose, to the contrary, that $0 < \beta'' < \beta$ holds. Then, we have $-\chi_j \in \mathrm{inc}(x + \beta(\chi_i + \chi_j), x + \beta'\chi_i + \beta''\chi_j)$. Therefore, the property (NNS$'$) implies that there exists some $\eta \in \mathbf{Z}$ with $0 < \eta \le \beta - \beta'' < \beta$ such that either $x + (\beta - \eta)(\chi_i + \chi_j) \in \mathcal{F}$ or $x + \beta\chi_i + (\beta - \eta)\chi_j \in \mathcal{F}$ (or both). By (10), we have $x + \beta\chi_i + (\beta - \eta)\chi_j \in \mathcal{F}$. It follows from Lemma A.1 applied to x and $x + \beta\chi_i + (\beta - \eta)\chi_j$ that there exists $\gamma \in \mathbf{Z}$ such that $0 \le \gamma \le \beta$, $\beta - \gamma \le \beta - \eta$, and $x + \gamma\chi_i \in \mathcal{F}$, a contradiction to (9).

Lemma A.5. *Let $z \in \mathcal{F}$.*
(i) *If $0 \le z(i) - x(i) \le \beta$ then $z(i) - x(i) \in \{0, \beta\}$.*
(ii) *If $0 \le z(j) - x(j) \le \beta$ then $z(j) - x(j) \in \{0, \beta\}$.*

Proof. [Proof of (i)] Suppose, to the contrary, that there exists some $z \in \mathcal{F}$ such that $x(i) < z(i) < x(i) + \beta$. Since $-\chi_i \in \mathrm{inc}(x + \beta(\chi_i + \chi_j), z)$, there exist some $s \in \mathrm{inc}(x + \beta(\chi_i + \chi_j), z) \cup \{\mathbf{0}\} \setminus \{-\chi_i\}$ and $\gamma \in \mathbf{Z}_{++}$ such that $\beta - \gamma \ge z(i) - x(i) > 0$ and

$$x' \equiv x + (\beta - \gamma)\chi_i + \beta\chi_j + \gamma s \in \mathcal{F}.$$

By (10), we have $s \ne -\chi_j$.

Suppose that $s = +\chi_j$. Then, we have $x + (\beta - \gamma)\chi_i + (\beta + \gamma)\chi_j \in \mathcal{F}$. Since $\beta - \gamma > 0$, it holds that $+\chi_i \in \mathrm{inc}(x, x + (\beta - \gamma)\chi_i + (\beta + \gamma)\chi_j)$. Hence, (NNS$'$) implies that there exists some $\eta \in \mathbf{Z}_{++}$ such that $\eta \le \beta - \gamma < \beta$ and either $x + \eta\chi_i \in \mathcal{F}$ or $x + \eta(\chi_i + \chi_j) \in \mathcal{F}$ (or both). This, however, is a contradiction to (9) or (10). Hence, we have $s \notin \{+\chi_j, -\chi_j\}$.

Suppose that $\gamma < \beta - \gamma$. Then, Lemma A.2 applied to x and x' implies that there exists some $\alpha'_1, \alpha'_2 \in \mathbf{Z}$ such that $0 \le \alpha'_1 \le \beta - \gamma$, $0 \le \alpha'_2 \le \beta$, $(\beta - \gamma - \alpha'_1) + (\beta - \alpha'_2) \le \gamma$, and $x + \alpha'_1\chi_i + \alpha'_2\chi_j \in \mathcal{F}$, a contradiction to Lemma A.4 since

$$\alpha'_1 \ge (\beta - \gamma) + (\beta - \alpha'_2) - \gamma \ge \beta - 2\gamma > 0.$$

Hence, we have $\gamma \geq \beta - \gamma$.

Since $+\chi_i \in \mathrm{inc}(x, x')$, there exists some $t \in \mathrm{inc}(x, x') \cup \{\mathbf{0}\} \setminus \{+\chi_i\} = \{+\chi_j, s, \mathbf{0}\}$ and $\eta \in \mathbf{Z}_{++}$ such that $x + \eta(\chi_i + t)$ is a neighbor of x and between x and x'. We have $\eta \leq \beta - \gamma < \beta$, and therefore it holds that $t = s$ by Lemma A.4, i.e. $x + \eta(\chi_i + s) \in \mathcal{F}$.

Lemma A.2 applied to $x + \beta(\chi_i + \chi_j)$ and $x + \eta(\chi_i + s)$ implies that there exists some $\mu_1, \mu_2 \in \mathbf{Z}$ such that $\eta \leq \mu_1 \leq \beta$, $0 \leq \mu_2 \leq \beta$, $(\mu_1 - \eta) + \mu_2 \leq \eta$, and $x + \mu_1\chi_i + \mu_2\chi_j \in \mathcal{F}$. Since $\mu_1 + \mu_2 \leq 2\eta < 2\beta$ and $\mu_1 \geq \eta > 0$, we have $(\mu_1, \mu_2) \notin \{(0, 0), (0, \beta), (\beta, \beta)\}$, a contradiction to Lemma A.4. This concludes the proof of (i).

[Proof of (ii)] Suppose, to the contrary, that there exists some $z \in \mathcal{F}$ such that $x(j) < z(j) < x(j) + \beta$. Since $-\chi_j \in \mathrm{inc}(x + \beta(\chi_i + \chi_j), z)$, there exist some $s \in \mathrm{inc}(x + \beta(\chi_i + \chi_j), z) \cup \{\mathbf{0}\} \setminus \{-\chi_j\}$ and $\gamma \in \mathbf{Z}_{++}$ such that $\beta - \gamma \geq z(j) - x(j) > 0$ and

$$x'' \equiv x + \beta\chi_i + (\beta - \gamma)\chi_j + \gamma s \in \mathcal{F}.$$

By (10), we have $s \neq -\chi_i$.

Suppose that $s = +\chi_i$. Then, we have $x + (\beta + \gamma)\chi_i + (\beta - \gamma)\chi_j \in \mathcal{F}$. Since $\beta - \gamma > 0$, it holds that $+\chi_j \in \mathrm{inc}(x, x + (\beta + \gamma)\chi_i + (\beta - \gamma)\chi_j)$. Hence, (NNS$'$) implies that there exists some $\eta \in \mathbf{Z}_{++}$ such that $\eta \leq \beta - \gamma < \beta$ and either $x + \eta\chi_j \in \mathcal{F}$ or $x + \eta(\chi_i + \chi_j) \in \mathcal{F}$ (or both). This, however, is a contradiction to Lemma A.4. Hence, we have $s \notin \{+\chi_i, -\chi_i\}$.

By Lemma A.2 applied to x and x'', there exist some integers α''_1 and α''_2 such that $0 \leq \alpha''_1 \leq \beta$, $0 \leq \alpha''_2 \leq \gamma < \beta$, $(\beta - \alpha''_1) + (\gamma - \alpha''_2) \leq \beta - \gamma$, and $x + \alpha''_1\chi_i + \alpha''_2 s \in \mathcal{F}$. Note that

$$\beta \geq \alpha''_1 \geq -(\beta - \gamma) + \beta + (\gamma - \alpha'_2) \geq \gamma > 0.$$

By the statement (i) shown above, we have $\alpha''_1 = \beta$, i.e., $x + \beta\chi_i + \alpha''_2 s \in \mathcal{F}$. By Lemma A.1 applied to x and $x + \beta\chi_i + \alpha''_2 s$, there exists some $\varepsilon \in \mathbf{Z}$ such that $0 \leq \varepsilon \leq \beta$, $\beta - \varepsilon \leq \alpha''_2 < \beta$, and $x + \varepsilon\chi_i \in \mathcal{F}$. Since $\varepsilon > 0$, this is a contradiction to (9).

By Lemma A.5, we have

$$\beta = \pi_i(\tilde{x}(i) + 1) - \pi_i(\tilde{x}(i)) = \pi_j(\tilde{x}(j) + 1) - \pi_j(\tilde{x}(j)). \tag{11}$$

Since $x + \beta(\chi_i + \chi_j) \in \mathcal{F}$, the equation (11) implies that $\tilde{x} + \chi_i + \chi_j \in \mathcal{J}$ and $+\chi_j \in \mathrm{inc}(\tilde{x}, \tilde{y})$. That is, we have (6) with $q = +\chi_j$. This concludes the proof of Theorem 3.1.

A.2 Proof of Lemma 3.1 (ii)

Let $x, y \in \mathcal{F}$, and put $\tilde{x} = \pi^{-1}(x), \tilde{y} = \pi^{-1}(y)$. Note that $\tilde{x}, \tilde{y} \in \mathcal{J}$.

Proof of "if" part We firstly show that if $\tilde{y}$ is a proper neighbor of $\tilde{x}$ in $\mathcal{J}$, then y is a proper neighbor of x in $\mathcal{F}$.

[Case 1: $|\{e \in E \mid \tilde{x}(e) \neq \tilde{y}(e)\}| = 1$] Let $i \in E$ be the unique element in $\{e \in E \mid \tilde{x}(e) \neq \tilde{y}(e)\}$. We may assume that $\tilde{x}(i) < \tilde{y}(i)$. Then, we have either

$$\text{(a) } \tilde{y} = \tilde{x} + \chi_i \in \mathcal{J}, \qquad \text{or} \qquad \text{(b) } \tilde{y} = \tilde{x} + 2\chi_i \in \mathcal{J} \text{ and } \tilde{x} + \chi_i \notin \mathcal{J}.$$

If (a) holds, then we have $y = x + \alpha_1\chi_i \in \mathcal{F}$ with $\alpha_1 = \pi_i(\tilde{x}(i)+1) - \pi_i(\tilde{x}(i))$, which is a proper neighbor of x in $\mathcal{F}$. If (b) holds, then we have $y = x + \alpha_2\chi_i \in \mathcal{F}$ with $\alpha_2 = \pi_i(\tilde{x}(i)+2) - \pi_i(\tilde{x}(i))$ and $x + \alpha_1\chi_i \notin \mathcal{F}$, implying that y is a proper neighbor of x in $\mathcal{F}$.

[Case 2: $|\{e \in E \mid \tilde{x}(e) \neq \tilde{y}(e)\}| = 2$] We may assume, without loss of generality, that $\tilde{y} = \tilde{x} + \chi_i + \chi_j$ for some distinct $i, j \in E$. Since $\tilde{y}$ is a proper neighbor of $\tilde{x}$ in $\mathcal{J}$, we may also assume that $\tilde{x} + \chi_i \notin \mathcal{J}$. Then, we have $y = x + \alpha\chi_i + \beta\chi_j \in \mathcal{F}$ and $x + \alpha\chi_i \notin \mathcal{F}$, where $\alpha = \pi_i(\tilde{x}(i)+1) - \pi_i(\tilde{x}(i))$, and $\beta = \pi_j(\tilde{x}(j)+1) - \pi_j(\tilde{x}(j))$. By the assumption and the definitions of α, β, we have

$$\text{if } x + \alpha'\chi_i + \beta'\chi_j \in \mathcal{F},\ 0 \le \alpha' \le \alpha,\ 0 \le \beta' \le \beta,\ \text{then } (\alpha', \beta') \in \{(0,0), (0,\beta), (\alpha,\beta)\}. \tag{12}$$

Hence, y is a proper neighbor of x in $\mathcal{F}$ if $\alpha = \beta$.

Suppose, to the contrary, that $\alpha \neq \beta$. If $\alpha > \beta$, then Lemma A.1 applied to x and $x + \alpha\chi_i + \beta\chi_j$ implies that there exists some $\alpha' \in \mathbf{Z}$ such that $0 \le \alpha' \le \alpha$, $\alpha - \alpha' \le \beta$, and $x + \alpha'\chi_i \in \mathcal{F}$. Since $\alpha > \beta$, we have $\alpha \ge \alpha' \ge \alpha - \beta > 0$. This, however, is a contradiction to (12). Hence, we have $\alpha < \beta$. Since $-\chi_j \in \mathrm{inc}(x + \alpha\chi_i + \beta\chi_j, x)$, the property (NNS$'$) implies that there exists some $\delta \in \mathbf{Z}_{++}$ such that either (a) $\delta \le \beta$ and $x + \alpha\chi_i + (\beta - \delta)\chi_j \in \mathcal{F}$, or (b) $\delta \le \min\{\alpha, \beta\}$ and $x + (\alpha - \delta)\chi_i + (\beta - \delta)\chi_j \in \mathcal{F}$ (or both). In either case, we have a contradiction to (12).

Proof of "only if" part We then show that if y is a proper neighbor of x in $\mathcal{F}$, then $\tilde{y}$ is a proper neighbor of $\tilde{x}$ in $\mathcal{J}$.

[Case 1: $|\{e \in E \mid x(e) \neq y(e)\}| = 1$] Let $i \in E$ be the unique element in $\{e \in E \mid x(e) \neq y(e)\}$. We may assume that $x(i) < y(i)$. Then, there exists some $\alpha_* \in \mathbf{Z}_{++}$ such that

$$y = x + \alpha_*\chi_i \in \mathcal{F}, \qquad x + \alpha'\chi_i \notin \mathcal{F} \quad (0 < \forall\alpha' < \alpha_*).$$

If $\alpha_* = \pi_i(\tilde{x}(i)+1) - \pi_i(\tilde{x}(i))$, then $\tilde{y} = \pi^{-1}(x + \alpha_*\chi_i) = \tilde{x} + \chi_i \in \mathcal{J}$, which is a proper neighbor of $\tilde{x}$ in $\mathcal{J}$ since $+\chi_i \in \mathrm{inc}(\tilde{x}, \tilde{y})$. Hence, suppose that $\alpha_* > \pi_i(\tilde{x}(i)+1) - \pi_i(\tilde{x}(i))$. Then, Lemma A.3 implies that $\alpha_* = \pi_i(\tilde{x}(i)+2) - \pi_i(\tilde{x}(i))$. Therefore, it holds that

$$\tilde{y} = \pi^{-1}(x + \alpha_*\chi_i) = \tilde{x} + 2\chi_i \in \mathcal{J}, \qquad \tilde{x} + \chi_i \notin \mathcal{J},$$

which shows that $\tilde{y}$ is a proper neighbor of $\tilde{x}$ in $\mathcal{J}$.

[Case 2: $|\{e \in E \mid x(e) \neq y(e)\}| = 2$] We may assume, without loss of generality, that $y = x + \alpha(\chi_i + \chi_j)$ for some distinct $i, j \in E$ and $\alpha \in \mathbf{Z}_{++}$. Since y is a proper neighbor of x in $\mathcal{F}$, we may also assume that

$$x + \alpha'\chi_i \notin \mathcal{F} \qquad (0 < \forall\alpha' < \alpha).$$

Then, in the same way as in the proof of Theorem 3.1, we can show that

$$\alpha = \pi_i(\tilde{x}(i) + 1) - \pi_i(\tilde{x}(i)) = \pi_j(\tilde{x}(j) + 1) - \pi_j(\tilde{x}(j))$$

(cf. (11)). This implies $\tilde{y} = \pi^{-1}(x+\alpha(\chi_i+\chi_j)) = \tilde{x}+\chi_i+\chi_j \in \mathcal{J}$ and $\tilde{x}+\chi_i \notin \mathcal{J}$. Therefore, $\tilde{y}$ is a proper neighbor of $\tilde{x}$ in $\mathcal{J}$.

A.3 Proof of Lemma 4.1

We prove the claim by induction on k. Since the case where $k = 1$ is obvious, we assume $k > 1$. By the induction hypothesis, there exists some $x \in \mathcal{F}$ such that

$$x(V_t \cap A, V_t \cap B) = \rho(V_t \cap A, V_t \cap B) \qquad (t = 1, 2, \dots, k-1).$$

Let $y \in \mathcal{F}$ be a vector satisfying $y(V_k \cap A, V_k \cap B) = \rho(V_k \cap A, V_k \cap B)$, and assume that y minimizes the value $||y-x||_1$ among all such y. We will show that y satisfies

$$y(V_t \cap A, V_t \cap B) = \rho(V_t \cap A, V_t \cap B) \qquad (t = 1, 2, \dots, k).$$

Assume, to the contrary, that there exists some $t \in \{1, 2, \dots, k-1\}$ such that $y(V_t \cap A, V_t \cap B) < \rho(V_t \cap A, V_t \cap B)$. Since $x(V_t \cap A, V_t \cap B) = \rho(V_t \cap A, V_t \cap B)$, we have either (1) $\{e \in E \mid e \in V_t \cap A,\ x(e) > y(e)\} \neq \emptyset$ or $\{e \in E \mid e \in V_t \cap B,\ x(e) < y(e)\} \neq \emptyset$ (or both). We consider the former case only since the latter case can be dealt with in a similar way.

Let $i \in E$ be an element such that $i \in V_t \cap A$ and $x(i) > y(i)$. Since $+\chi_i \in \mathrm{inc}(y, x)$, the property (NNS$'$) implies that there exist $q \in \mathrm{inc}(y, x) \cup \{\mathbf{0}\} \setminus \{+\chi_i\}$ and $\alpha \in \mathbf{Z}_{++}$ such that $y' = y + \alpha(\chi_i + q) \in \mathcal{F}$ and y' is between y and x. If $q = \mathbf{0}$, then we have

$$y'(V_k \cap A, V_k \cap B) > y(V_k \cap A, V_k \cap B) = \rho(V_k \cap A, V_k \cap B)$$

since $i \in V_t \cap A \subseteq V_k \cap A$, a contradiction. If $q \neq \mathbf{0}$, then we have

$$y'(V_k \cap A, V_k \cap B) \geq y(V_k \cap A, V_k \cap B) = \rho(V_k \cap A, V_k \cap B),$$

and the inequality must hold with equality by the definition of ρ. In addition, it holds that $||y' - x||_1 < ||y - x||_1$, a contradiction of the choice of y.

A.4 Proof of Lemma 4.3

By Lemma 4.2 it suffices to show that ρ satisfies the conditions (1) and (2).

We firstly show the condition (1). By Lemma 4.1, there exists a vector $x \in \mathcal{F}$ satisfying

$$\begin{aligned} x((X \cap Y) \cap A, (X \cap Y) \cap B) &= \rho((X \cap Y) \cap A, (X \cap Y) \cap B), \\ x(X \cap A, X \cap B) &= \rho(X \cap A, X \cap B), \\ x((X \cup Y) \cap A, (X \cup Y) \cap B) &= \rho((X \cup Y) \cap A, (X \cup Y) \cap B), \end{aligned}$$

which implies the desired inequality as follows:

$$\begin{aligned} &\rho((X \cup Y) \cap A, (X \cup Y) \cap B) + \rho((X \cap Y) \cap A, (X \cap Y) \cap B) \\ &= x((X \cup Y) \cap A, (X \cup Y) \cap B) + x((X \cap Y) \cap A, (X \cap Y) \cap B) \\ &= x(X \cap A, X \cap B) + x(Y \cap A, Y \cap B) \\ &\leq \rho(X \cap A, X \cap B) + \rho(Y \cap A, Y \cap B). \end{aligned}$$

We then show the condition (2). By Lemma 4.1, there exists a vector $x \in \mathcal{F}$ satisfying

$$x(X) - x(Y) = \rho(X, Y), \qquad x(X) - x(Y) - x(i) = \rho(X, Y \cup \{i\}),$$

which implies the desired inequality as follows:

$$\begin{aligned} \rho(X \cup \{i\}, Y) + \rho(X, Y \cup \{i\}) &\geq \{x(X) + x(i) - x(Y)\} + \{x(X) - x(Y) - x(i)\} \\ &= 2\{x(X) - x(Y)\} = \rho(X, Y). \end{aligned}$$

A.5 Proof of Lemma 4.6

We define the vectors $\ell, u \in \mathbf{Z}^E$ by $\ell(e) = \min\{x(e), y(e)\}$ and $u(e) = \max\{x(e), y(e)\}$ for $e \in E$. Since $x \in \mathcal{F} \cap [\ell, u]$, the set $\mathcal{F} \cap [\ell, u]$ is nonempty, and therefore its convex closure $S = \mathrm{conv}(\mathcal{F} \cap [\ell, u])$ is an integral bisubmodular polyhedron. By the definitions of ℓ and u, the vector x is an extreme point of S. We consider the tangent cone $\mathrm{TC}(x)$ of S at x, which is given by $\mathrm{TC}(x) = \{\alpha z \mid x \in \mathbf{R}^E,\ x + z \in S,\ \alpha \in \mathbf{R},\ \alpha \geq 0\}$.

Lemma A.6. *There exists an extreme ray $d \in \mathbf{R}^E$ of $\mathrm{TC}(x)$ that is a positive multiple of either $+\chi_i$, $+\chi_i + \chi_k$, or $+\chi_i - \chi_k$ for some $k \in E \setminus \{i\}$.*

Proof. By the definition of the tangent cone, we have $y - x \in \mathrm{TC}(x)$. Since $x(i) > y(i)$, there exists some extreme ray $d \in \mathbf{R}^E$ of $\mathrm{TC}(x)$ such that $d(i) > 0$. Since S is a bisubmodular polyhedron, every extreme ray of the tangent cone $\mathrm{TC}(x)$ is a positive multiple of a vector $+\chi_j, -\chi_j, +\chi_j + \chi_k, +\chi_j - \chi_k$, or $-\chi_j - \chi_k$ for some $j, k \in E$ (see [1, Theorem 3.5]). Hence, the extreme ray d is a positive multiple of either $+\chi_i$, $+\chi_i + \chi_k$, or $+\chi_i - \chi_k$ for some $k \in E \setminus \{i\}$.

Proof (Proof of Lemma 4.6). By Lemma A.6, there exists an extreme ray $d \in \mathbf{R}^E$ of $\mathrm{TC}(x)$ that is a positive multiple of either $+\chi_i$, $+\chi_i + \chi_k$, or $+\chi_i - \chi_k$ for some $k \in E \setminus \{i\}$. Since d is an extreme ray of $\mathrm{TC}(x)$, there exists some vector $z_0 \in S$ such that z_0 is an extreme point of S and $z_0 - x$ is a positive multiple of d. Since $d(i) > 0$, we have $z_0(i) > x(i)$. The vector z_0 is contained in $\mathcal{F} \cap [\ell, u]$ since it is an extreme point of $S = \mathrm{conv}(\mathcal{F} \cap [\ell, u])$. This implies, in particular, $z_0 \in \mathcal{F}$ and z_0 is between x and y.

A.6 Proof of Theorem 4.1 for Infinite Neighbor Systems

We prove Theorem 4.1 for the case where a neighbor system $\mathcal{F}$ is not necessarily a finite set.

Let $x_0 \in \mathbf{Z}^E$ be any vector in $\mathcal{F}$ and define $\mathcal{F}_k$ for $k = 1, 2, \ldots$ by

$$\mathcal{F}_k = \{x \in \mathcal{F} \mid |x(e) - x_0(e)| \le k \ (\forall e \in E)\}.$$

We also define a function $\rho_k : 3^E \to \mathbf{R} \cup \{+\infty\}$ $(k = 1, 2, \ldots)$ by

$$\rho_k(X, Y) = \max\{x(X) - x(Y) \mid x \in \mathcal{F}_k\} \qquad ((X, Y) \in 3^E).$$

By Proposition 2.1 (iii), each $\mathcal{F}_k$ is a neighbor system, and therefore ρ_k is a bisubmodular function by Lemma 4.3. We note that $\mathcal{F}_k$ is a finite set and therefore ρ_k takes finite values. Moreover, it holds that $\lim_{k\to+\infty} f_k(X, Y) = f(X, Y)$ for every $(X, Y) \in 3^E$. Therefore, we have

$$\begin{aligned}
\rho(X_1, Y_1) + \rho(X_2, Y_2) &= \lim_{k\to+\infty} \rho_k(X_1, Y_1) + \lim_{k\to+\infty} \rho_k(X_2, Y_2) \\
&\ge \lim_{k\to+\infty} \rho_k((X_1 \cup X_2) \setminus (Y_1 \cup Y_2), (Y_1 \cup Y_2) \setminus (X_1 \cup X_2)) \\
&\qquad + \lim_{k\to+\infty} \rho_k(X_1 \cap X_2, Y_1 \cap Y_2) \\
&= \rho((X_1 \cup X_2) \setminus (Y_1 \cup Y_2), (Y_1 \cup Y_2) \setminus (X_1 \cup X_2)) \\
&\qquad + \rho(X_1 \cap X_2, Y_1 \cap Y_2) \qquad (\forall (X_1, Y_1), (X_2, Y_2) \in 3^E),
\end{aligned}$$

i.e., ρ is also a bisubmodular function.

We then show that $\mathrm{conv}(\mathcal{F}) = \mathrm{P}_*(\rho)$ holds. Since $\mathcal{F} \subseteq \mathrm{P}_*(\rho)$, we have $\mathrm{conv}(\mathcal{F}) \subseteq \mathrm{P}_*(\rho)$. It holds that $\mathrm{P}_*(\rho) = \lim_{k\to+\infty} \mathrm{P}_*(\rho_k)$, and that $\mathrm{P}_*(\rho_k) = \mathrm{conv}(\mathcal{F}_k)$ for each $k = 1, 2, \ldots$ by Lemma 4.5. Since $\mathrm{conv}(\mathcal{F}_k) \subseteq \mathrm{conv}(\mathcal{F})$, we have

$$\mathrm{P}_*(\rho) = \lim_{k\to+\infty} \mathrm{P}_*(\rho_k) = \lim_{k\to+\infty} \mathrm{conv}(\mathcal{F}_k) \subseteq \mathrm{conv}(\mathcal{F}).$$

Hence, $\mathrm{conv}(\mathcal{F}) = \mathrm{P}_*(\rho)$ holds.

A.7 Proof of Theorem 5.5

Computation of all proper neighbors of x can be done as follows. We firstly compute proper neighbors of the form $x + \alpha\chi_i$ $(i \in E,\ 0 < \alpha \le k)$. If the set $\{\alpha \mid x + \alpha\chi_i \in \mathcal{F},\ 0 < \alpha \leq k\}$ is nonempty, then we compute the value $\alpha_i^+ = \min\{\alpha \mid x + \alpha\chi_i \in \mathcal{F},\ 0 < \alpha \leq k\}$, and output $x + \alpha_i^+\chi_i$, which is a proper neighbor of x. Otherwise, we put $\alpha_i^+ = +\infty$. It is easy to see that this can be done in $\mathrm{O}(nk)$ time. Similarly, we can compute all proper neighbors of the form $x - \alpha\chi_i$ and values $\alpha_i^- = \min\{\alpha \mid x - \alpha\chi_i \in \mathcal{F},\ 0 < \alpha \leq k\}$ in $\mathrm{O}(nk)$ time.

We then compute proper neighbors of the form $x+\alpha(\chi_i+\chi_j)$ for some distinct $i, j \in E$ and $\alpha \in \mathbf{Z}_{++}$ with $\alpha \leq k$. If the set $\{\alpha \mid x+\alpha(\chi_i+\chi_j) \in \mathcal{F},\ 0 < \alpha \leq k\}$ is nonempty, then we compute the value α_{ij}^+ defined by $\alpha_{ij}^+ = \min\{\alpha \mid x+\alpha(\chi_i + \chi_j) \in \mathcal{F},\ 0 < \alpha \leq k\}$. If $\alpha_{ij}^+ < \alpha_i^+$ or $\alpha_{ij}^+ < \alpha_j^+$, then the vector $x + \alpha_{ij}^+\chi_{ij}$ is a proper neighbor, and output it. It is easy to see that this can be done in $\mathrm{O}(n^2k)$ time. Similarly, we can compute all proper neighbors of the forms $x+\alpha(\chi_i-\chi_j)$ and $x+\alpha(-\chi_i-\chi_j)$ in $\mathrm{O}(n^2k)$ time. Hence, we can compute all proper neighbors of x in $\mathrm{O}(n^2k)$ time.

A.8 Time Complexity Analysis of Algorithm Greedy

We analyze the running time of the algorithm Greedy in Section 5.2. For each $e \in E$, the values $a_{\mathcal{F}}(e)$ and $b_{\mathcal{F}}(e)$ can be computed in $\mathrm{O}(n^2k\log\Phi(\mathcal{F}))$ time by using the algorithm for linear optimization by Hartvigsen [14]. Hence, Step 0 can be done in $\mathrm{O}(n^3k\log\Phi(\mathcal{F}))$ time. Steps 1 and 2 can be done in $\mathrm{O}(n^2k)$ time by Theorem 5.5. The values α_-, α_+ can be also computed in $\mathrm{O}(n^2k)$ time by using the following property and Theorem 5.5.

Proposition A.1. *Let $\mathcal{F} \subseteq \mathbf{Z}^E$ be an all-neighbor system and $x \in \mathcal{F}$.*
(i) *Suppose that $\{y \in \mathcal{F} \mid y(i) < x(i)\} \neq \emptyset$. Then, there exists a proper neighbor $y_* \in \mathcal{F}$ of x such that $x(i) - y_*(i) = \min\{x(i) - y(i) \mid y \in \mathcal{F},\ y(i) < x(i)\}$.*
(ii) *Suppose that $\{y \in \mathcal{F} \mid y(i) > x(i)\} \neq \emptyset$. Then, there exists a proper neighbor $y_* \in \mathcal{F}$ of x such that $y_*(i) - x(i) = \min\{y(i) - x(i) \mid y \in \mathcal{F},\ y(i) > x(i)\}$.*

Proof. We prove (i) only. Let $z \in \mathcal{F}$ be a vector such that $x(i)-z(i) = \min\{x(i)-y(i) \mid y \in \mathcal{F},\ y(i) < x(i)\}$. Since $-\chi_i \in \mathrm{inc}(x, z)$, the property (NNS$'$) implies that there exists some $q \in \mathrm{inc}(x, z) \cup \{\mathbf{0}\} \setminus \{-\chi_i\}$ and $\alpha \in \mathbf{Z}_{++}$ such that $x' \equiv x + \alpha(-\chi_i + q)$ is a neighbor of x and x' is between x and y_0.

If x' is a proper neighbor of x, then the choice of z implies $y_*(i) = z(i)$ since $x(i) > x'(i) \geq z(i)$. Suppose that x' is not a proper neighbor of x. Then, $q = \mathbf{0}$ and there exists some $\alpha' \in \mathbf{Z}_{++}$ with $\alpha' \leq \alpha$ such that $x'' \equiv x - \alpha'\chi_i$ is a proper neighbor of x. By the choice of z, we have $x''(i) = z(i)$ since $x(i) > x''(i) = x(i) - \alpha' \geq z(i)$.

This property implies that Step 3 can be done in $\mathrm{O}(n^2k)$ time.

To bound the number of iterations of the algorithm, we consider the value $||b - a||_1$. Suppose that we have $p_* = -\chi_i$ in Step 3, and denote by b_{old} (resp., b_{new}) the vector b before update (resp., after update). Then, it holds that $a(i) \leq b_{\mathrm{new}}(i) = x(i) - \alpha_- < x(i) \leq b_{\mathrm{old}}(i)$, implying that $||b_{\mathrm{new}} - a||_1 < ||b_{\mathrm{old}} - a||_1$. If $p_* = +\chi_i$ holds in Step 3, then we can show in the same way that $||b - a_{\mathrm{new}}||_1 < ||b - a_{\mathrm{old}}||_1$, where a_{old} (resp., a_{new}) the vector a before update (resp., after update). Hence, the value $||b - a||_1$ reduces in each iteration, and therefore the number of iterations is bounded by $n\,\Phi(\mathcal{F})$.

A.9 Proof of Theorem 5.7

To prove Theorem 5.7, we use the following property of jump systems.

Theorem A.1 ([20, Theorem 4.3]). *Let $\mathcal{J} \subseteq \mathbf{Z}^E$ be a jump system. Define a set $\mathcal{J}^\circ \subseteq \mathbf{Z}^E$ by $\mathcal{J}^\circ = \mathcal{J} \cap [a^\circ_{\mathcal{J}}, b^\circ_{\mathcal{J}}]$, where for each $e \in E$,*

$$a_{\mathcal{J}}(e) = \min\{x(e) \mid x \in \mathcal{J}\}, \qquad b_{\mathcal{J}}(e) = \max\{x(e) \mid x \in \mathcal{J}\},$$
$$a^\circ_{\mathcal{J}}(e) \ = \ a_{\mathcal{J}}(e) + \left\lfloor \tfrac{b_{\mathcal{J}}(e) - a_{\mathcal{J}}(e)}{n} \right\rfloor, \qquad b^\circ_{\mathcal{J}}(e) \ = \ b_{\mathcal{J}}(e) - \left\lfloor \tfrac{b_{\mathcal{J}}(e) - a_{\mathcal{J}}(e)}{n} \right\rfloor.$$

Then, the set $\mathcal{J}^\circ$ is nonempty.

Proof of (i) Let $\mathcal{F}$ be a given N_k-neighbor system, and define vectors $\ell, u \in \mathbf{Z}^E$, a jump system $\mathcal{J} \subseteq \mathbf{Z}^E$, and a family of strictly increasing functions $\pi_e : [\ell(e), u(e)] \to \mathbf{Z}$ ($e \in E$) as in Section 3. For $e \in E$, let $\ell'(e)$ be the minimum integer with $\pi_e(\ell'(e)) \geq a^\bullet_{\mathcal{F}}(e)$ and $u'(e)$ be the maximum integer with $\pi_e(u'(e)) \leq b^\bullet_{\mathcal{F}}(e)$. Then, we have

$$\mathcal{F}^\bullet = \{x \in \mathcal{F} \mid \pi_e(\ell'(e)) \leq x(e) \leq \pi_e(u'(e)) \ (\forall e \in E)\}.$$

Therefore, $\mathcal{F}^\bullet$ is nonempty if and only if the set $\mathcal{J}^\bullet \subseteq \mathcal{J}$ given by

$$\mathcal{J}^\bullet = \{x \in \mathcal{J} \mid \ell'(e) \leq x(e) \leq u'(e) \ (\forall e \in E)\}$$

is nonempty. By Theorem A.1, the set $\mathcal{J}^\bullet$ is nonempty if it holds that

$$\ell'(e) \leq \ell(e) + \left\lfloor \frac{u(e) - \ell(e)}{n} \right\rfloor, \quad u'(e) \geq u(e) - \left\lfloor \frac{u(e) - \ell(e)}{n} \right\rfloor \qquad (\forall e \in E). \tag{13}$$

Therefore, it suffices to show that the inequalities (13) hold. In the following, we prove the former inequality in (13) only since the latter can be proven in a similar way.

For $e \in E$, it holds that

$$(\ell'(e) - 1) - \ell(e) \leq \pi_e(\ell'(e) - 1) - \pi_e(\ell(e)) \leq (a^\bullet_{\mathcal{F}}(e) - 1) - a_{\mathcal{F}}(e),$$

where the first inequality follows from the fact that π_e is a strictly increasing function, and the last inequality is by the definition of $\ell'(e)$. Hence, we have

$$\ell'(e) - \ell(e) \leq a^\bullet_{\mathcal{F}}(e) - a_{\mathcal{F}}(e) = \left\lfloor \frac{b_{\mathcal{F}}(e) - a_{\mathcal{F}}(e)}{nk} \right\rfloor. \tag{14}$$

Lemma A.7. *It holds that $\pi_e(\alpha + 1) - \pi_e(\alpha) \leq k$ ($\forall e \in E$, $\ell(e) \leq \forall \alpha < u(e)$).*

Proof. Let $x \in \mathcal{F}$ (resp., $y \in \mathcal{F}$) be a vector with $x(e) = \pi_e(\alpha)$ (resp., $y(e) = \pi_e(\alpha + 1)$). Since $+\chi_e \in \mathrm{inc}(x, y)$, the property (NNS$'$) implies that there exist $q \in \mathrm{inc}(x, y) \cup \{\mathbf{0}\} \setminus \{+\chi_e\}$ and $\alpha \in \mathbf{Z}_{++}$ such that $x' \equiv x + \alpha(\chi_e + q)$ is a neighbor of x, $||x' - x||_1 \leq k$, and x' is between x and y. In particular, we have $x(e) < x'(e) \leq y(e)$. By the definition of the value $\pi_e(\alpha + 1)$, we have $x'(e) = \pi_e(\alpha + 1) = y(e)$. Hence, $\pi_e(\alpha + 1) - \pi_e(\alpha) = y(e) - x(e) \leq ||x' - x||_1 \leq k$.

It follows from Lemma A.7 that

$$b_{\mathcal{F}}(e) - a_{\mathcal{F}}(e) \leq k(u(e) - \ell(e)) \qquad (e \in E).$$

From this inequality and (14) follows that

$$\ell'(e) - \ell(e) \leq \left\lfloor \frac{b_{\mathcal{F}}(e) - a_{\mathcal{F}}(e)}{nk} \right\rfloor \leq \left\lfloor \frac{u(e) - \ell(e)}{n} \right\rfloor,$$

i.e., the former inequality in (13) holds. This concludes the proof of (i).

Proof of (ii) We can compute a vector in $\mathcal{F}^\bullet$ by the following algorithm.

Let x_0 be a given vector in $\mathcal{F}$ and assume for simplicity that $E = \{1, 2, \ldots, n\}$. For $i = 1, 2, \ldots, n$, we iteratively define a vector $x_i \in \mathcal{F}$ as follows:

• if $a^\bullet_{\mathcal{F}}(i) \leq x_{i-1}(i) \leq b^\bullet_{\mathcal{F}}(i)$, then set $x_i = x_{i-1}$.
• if $x_{i-1}(i) < a^\bullet_{\mathcal{F}}(i)$, then let x_i be a vector in $\mathcal{F}$ which maximizes the value $x_i(i)$ under the constraints $x_i(i) \leq b^\bullet_{\mathcal{F}}(i)$ and $a^\bullet_{\mathcal{F}}(e) \leq x_i(e) \leq b^\bullet_{\mathcal{F}}(e)$ $(e = 1, 2, \ldots, i-1)$. • if $x_{i-1}(i) > b^\bullet_{\mathcal{F}}(i)$, then let x_i be a vector in $\mathcal{F}$ which minimizes the value $x_i(i)$ under the constraints $x_i(i) \geq a^\bullet_{\mathcal{F}}(i)$ and $a^\bullet_{\mathcal{F}}(e) \leq x_i(e) \leq b^\bullet_{\mathcal{F}}(e)$ $(e = 1, 2, \ldots, i-1)$.

By the statement (i) of Theorem 5.7, we see that the set

$$\mathcal{F}_i \equiv \mathcal{F} \cap \{x \mid a^\bullet_{\mathcal{F}}(e) \leq x_i(e) \leq b^\bullet_{\mathcal{F}}(e) \ (e = 1, 2 \ldots, i)\}$$

is nonempty for all $i = 1, 2, \ldots, n$. Therefore, the vector x_i is contained in $\mathcal{F}_i$; in particular, we have $x_n \in \mathcal{F}_n = \mathcal{F}^\bullet$.

Each iteration of the algorithm above can be done by using the algorithm for linear optimization by Hartvigsen [14], which requires $\mathrm{O}(n^2 k \log \Phi(\mathcal{F}))$ time. Hence, a vector in $\mathcal{F}^\bullet$ can be found in $\mathrm{O}(n^3 k \log \Phi(\mathcal{F}))$ time.

A.10 Time Complexity Analysis of Algorithm Domain_Reduction

We analyze the number of iterations of the algorithm Domain_Reduction in Section 5.3. Denote by a_m, b_m the vectors a, b at the beginning of the m-th iteration. It is clear that the value $b_m(e) - a_m(e)$ is nonincreasing with respect to m for each $e \in E$.

We have $b_0(e) - a_0(e) \leq \Phi(\mathcal{F})$ for all $e \in E$ at the beginning of the algorithm, and if $b_m(e) - a_m(e) < 1$ for all $e \in E$, then we obtain an optimal solution immediately. Hence, it follows from Lemma 5.1 that the algorithm Domain_Reduction terminates in $\mathrm{O}(n^2 k \log \Phi(\mathcal{F}))$ iterations.

By Theorem 5.7 (ii), Step 1 can be done in $\mathrm{O}(n^3 k \log \Phi(\mathcal{F}))$ time. As shown in Section 5.2, Step 0 can be done in $\mathrm{O}(n^3 k \log \Phi(\mathcal{F}))$ time, and Steps 2, 3, and 4 can be done in $\mathrm{O}(n^2 k)$ time. Hence, we obtain the following theorem.

Finally, we give a proof of Lemma 5.1.

Proof (Proof of Lemma 5.1). We show the inequality in the case $p_* = -\chi_i$ only. Let $x \in (\mathcal{F} \cap [a_m, b_m])^\bullet$ be the vector chosen in Step 1 of the m-th iteration. Then,

$$\begin{aligned} b_{m+1}(i) - a_{m+1}(i) &= x(u) - \alpha_- - a_m(i) \leq \left(b_m(i) - \left\lfloor \tfrac{b_m(i) - a_m(i)}{nk} \right\rfloor\right) - 1 - a_m(i) \\ &< \left(1 - \tfrac{1}{nk}\right)(b_m(i) - a_m(i)). \end{aligned}$$

□