Introduction to Discrete Convex Analysis

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Discrete Convex Analysis

Discrete Convex Analysis [Murota 1996]
--- theoretical framework for discrete optimization problems

discrete analogue of
Convex Analysis
in continuous optimization

generalization of Theory of
Matroid/Submodular Function
in discrete opitmization

- key concept: two discrete convexity: L-convexity & M-convexity
  - generalization of Submodular Set Function & Matroid
- various nice properties
  - local optimal ←→ global optimal
  - duality theorem, separation theorem, conjugacy relation
- set/function are discrete convex → problem is tractable

Applications

- Combinatorial Optimization
  - matching, min-cost flow, shortest path, min-cost tension
- Math economics / Game theory
  - allocation of indivisible goods, stable marriage
- Operations research
  - inventory system, queueing, resource allocation
- Discrete structures
  - finite metric space
- Algebra
  - polynomial matrix, tropical geometry

History of Discrete Convex Analysis

1935: Matroid — Whitney

1965: Polymatroid, Submodular Function — Edmonds

1983: Submodularity and Convexity — Lovász, Frank, Fujishige

1992: Valuated Matroid — Dress, Wenzel

1996: Discrete Convex Analysis, L-/M-convexity — Murota

1996-2000: variants of L-/M-convexity — Fujishige, Murota, Shioura

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1971: discretely convex function — Miller

1990: integrally convex function — Favati-Tardella

Today's Talk

- fundamental properties of M-convex & L-convex functions
- comparison with other discrete convexity
  - convex-extensible fn
  - Miller's discretely convex fn
  - Favati-Tardella's integrally convex fn

Outline of Talk

- Overview of Discrete Convex Analysis
- Desirable Properties of Discrete Convexity
- convex-extensible fn
- Miller’s discretely convex fn
- Favati-Tardella’s integrally convex fn
- M-convex & L-convex fns
- duality and conjugacy theorems for discrete convex fn

Desirable Properties of Discrete Convexity

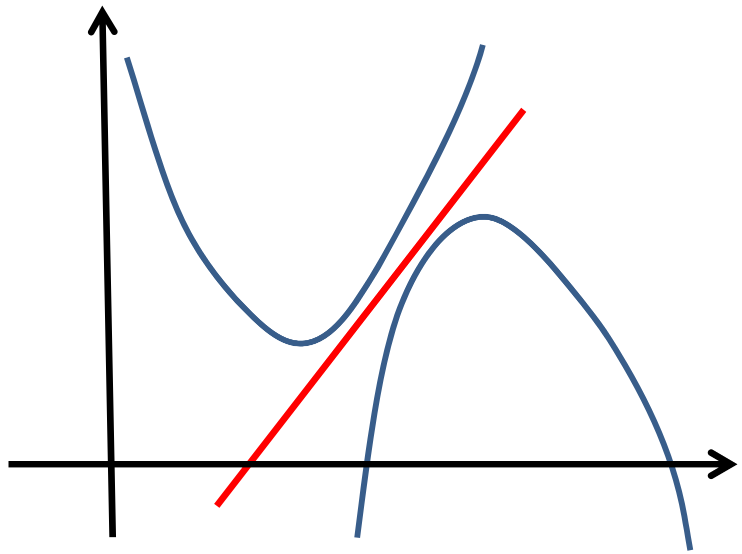
Important Properties of Convex Fn

- optimality condition by local property

  x: local minimum in some neighborhood ➔ global minimum

- conjugacy relationship
  - conjugate of convex fn → convex fn

- duality theorems
  - Fenchel duality
  - separation theorem



Desirable Properties of Discrete Convex Fn

- discrete convexity = “convexity” for functions $f: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$
  - convex extensibility
    - can be extended to convex fn on $\mathbb{R}^n$
  - optimality condition by local property
    - local minimum ➔ global minimum
      - local minimality depends on choice of neighborhood
  - duality theorems
    - “discrete” Fenchel duality
    - “discrete” separation theorem
  - conjugacy relationship
    - conjugate of “discrete” convex fn → “discrete” convex fn

Classes of Discrete Convex Fns

- convex-extensible fn
- discretely convex fn (Miller 1971)
- integrally convex fn (Favati-Tardella 1990)
- M-convex fn, L-convex fn (Murota 1995, 1996)

satisfy desirable properties?

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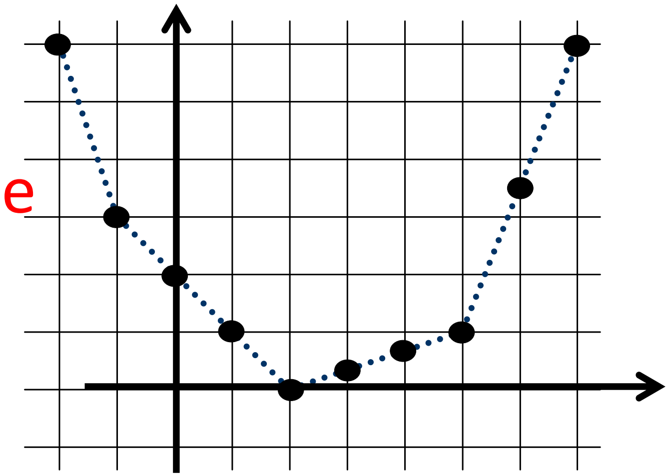
Convex-Extensible Function

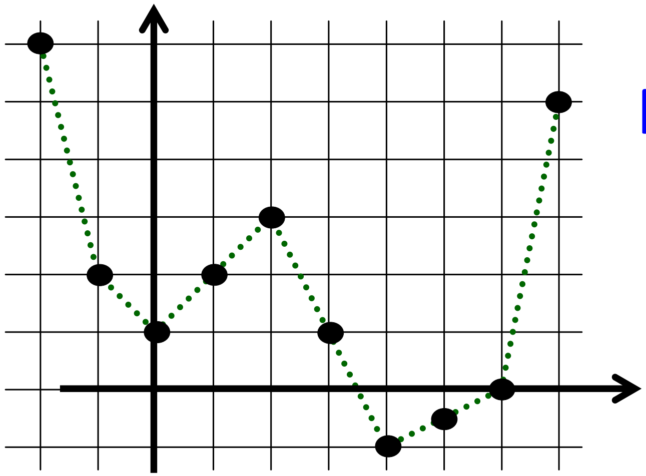
Definition of Convex-Extensible Fn

- a natural candidate for "discrete convexity"
- Def: $f: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is convex-extensible

⟺ $\exists \tilde{f}: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, convex fn s.t. $\tilde{f}(x) = f(x)$ $(\forall x \in \mathbb{Z}^n)$

convex -extensible





NOT convex -extensible

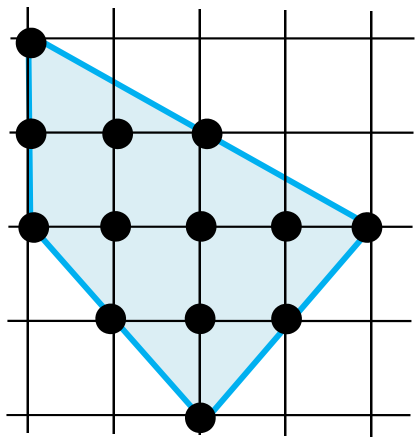
Definition of Convex-Extensible Set

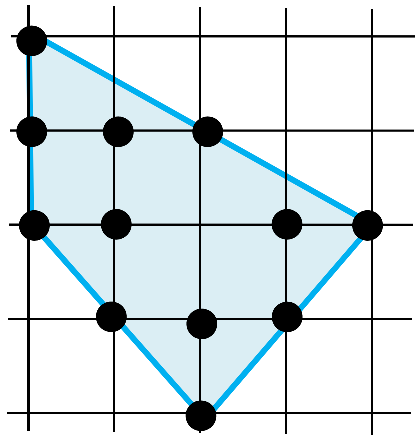
- Def: $S \subseteq \mathbb{Z}^n$ is convex-extensible

  ⇔ indicator fn $\delta_S: \mathbb{Z}^n \to \{0, +\infty\}$ is convex-extensible

  ⇔ $\mathrm{conv}(S) \cap \mathbb{Z}^n = S$ ("no-hole" condition)

convex -extensible



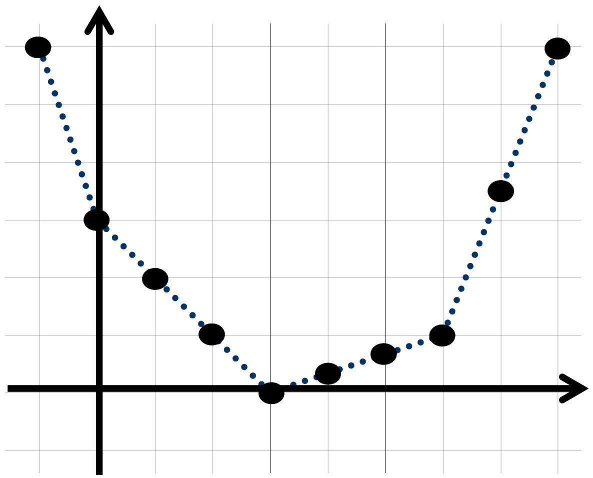


NOT convex -extensible

Properties of Convex-Extensible Fn

- if n=1, satisfies various nice properties
  - convex-extensible ←→ $f(x-1)+f(x+1) \geq 2f(x)$
  - local min=global min, conjugacy, duality, etc.

    → desirable concept as discrete convexity

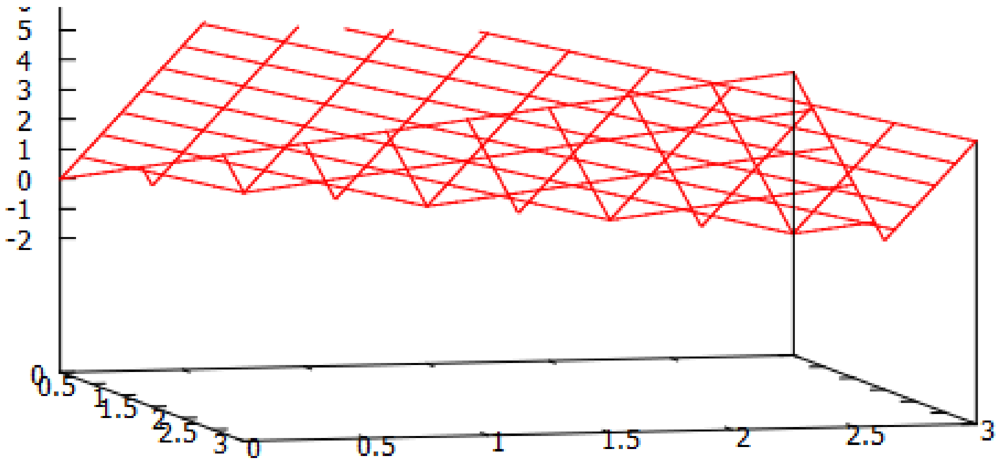


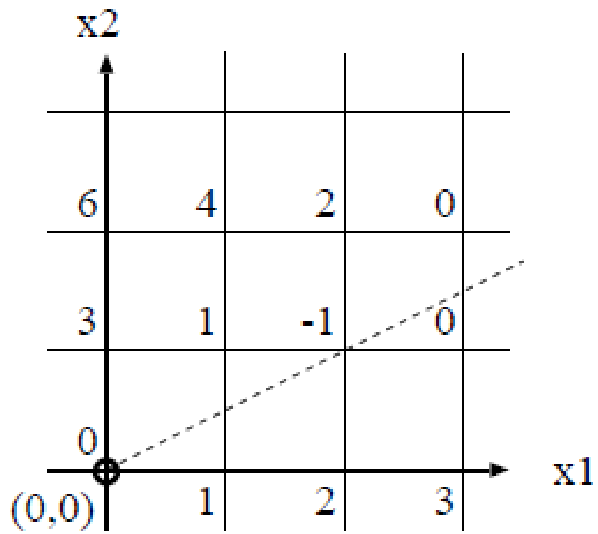
- if n≧2,
  - convex-extensible (by definition)
  - what else?

Bad Results of Conv.-Extensible Fn

- any function f with $\mathrm{dom}\ f = \{0,1\}^n$ is convex-extensible
  ➔ no good structure
- local opt ≠ global opt: $\forall k \in \mathbb{Z}_+, \exists f$: convex-extensible fn s.t.
  x: local min in $\{z \in \mathbb{Z}^n \mid ||z - x||_\infty \leq k\}$ but NOT global min

Example: $\mathrm{dom}\ f = \mathbb{Z}^2_+,\ f(x_1, x_2) = \max\{x_1 - 3x_2, -2x_1 + 3x_2\}$





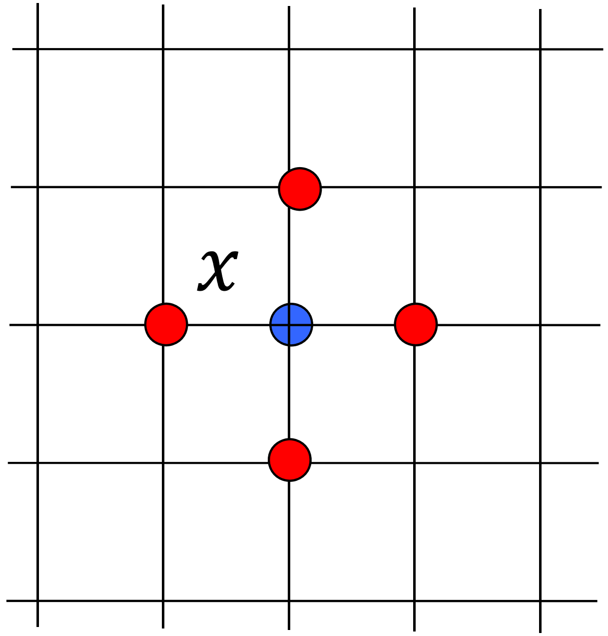
x=(0,0): local min in $\{z \in \mathbb{Z}^n \mid ||z - x||_\infty \leq 1\}$, f(0,0)>f(2,1)

Separable-Convex Function

- Def: $f: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is separable-convex ←→

$f(x) = \sum_{i=1}^{n} \varphi_i(x(i))$, each $\varphi_i: \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ is discrete convex

  - examples: $\sum_{i=1}^{n} x(i)^2$, $-\sum_{i=1}^{n} \log x(i)$, etc.
  - satisfy various nice properties
    - convex-extensible
    - local min w.r.t. $\{z \mid ||z - x||_1 \le 1\}$ = global min
  - but, function class is too small
    - e.g., dom f is integer interval



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Miller’s Discretely Convex Fn

Definition of Discretely Convex Fn

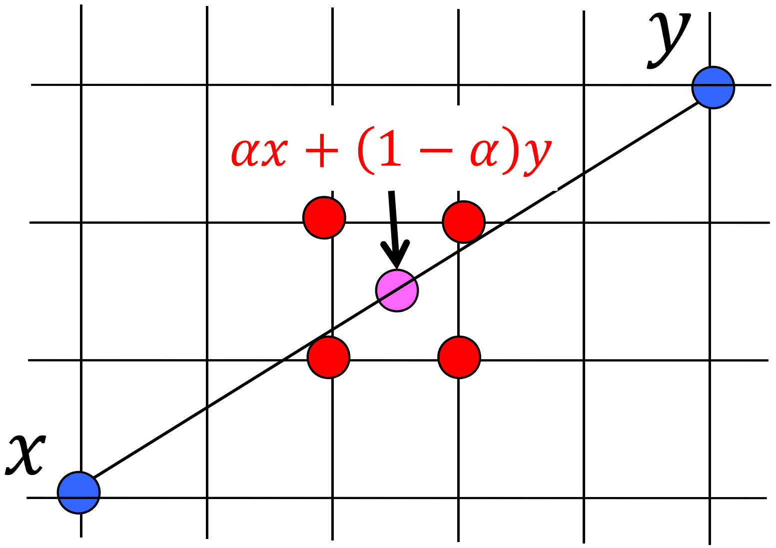
- defined by discretized version of convex inequality
- Def: $f: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is discretely convex (Miller 1971)

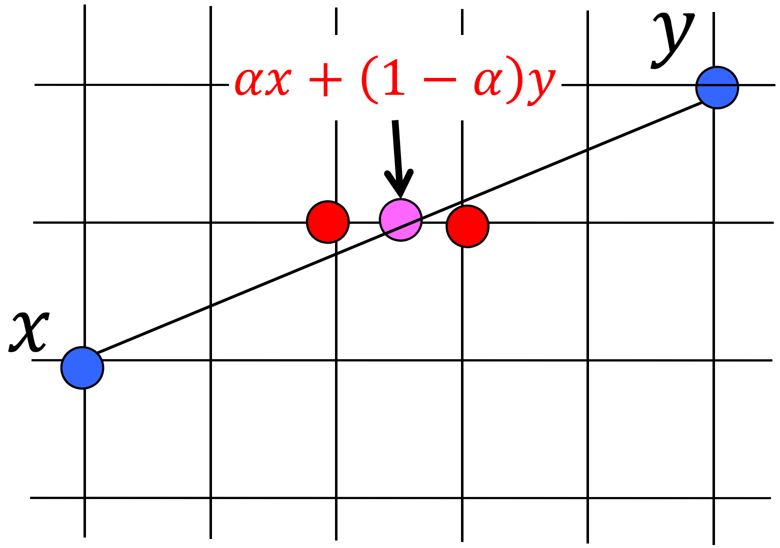
⇔ $\forall x, y \in \mathbb{Z}^n,\ \alpha \in [0,1], \quad s \equiv \alpha x + (1-\alpha) y$

~~$f(s)$~~ $\leq \alpha f(x) + (1-\alpha) f(y)$

$\min\{f(z) \mid z(i) = \lfloor s(i) \rfloor \ or \ \lceil s(i) \rceil \ \ (\forall i)\}$

Prop: $s \in \mathbb{Z}^n \Rightarrow f(s) \leq \alpha f(x) + (1-\alpha) f(y)$





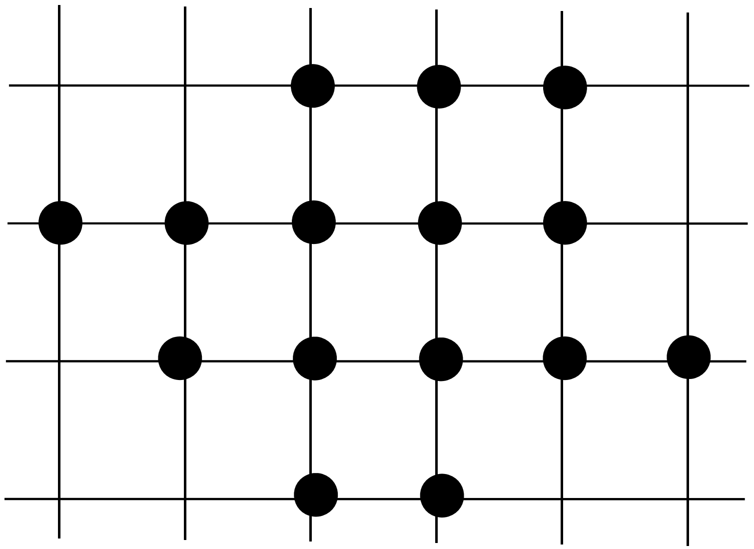
Definition of Discretely Convex Set

- Def: $S \subseteq \mathbb{Z}^n$ is discretely convex

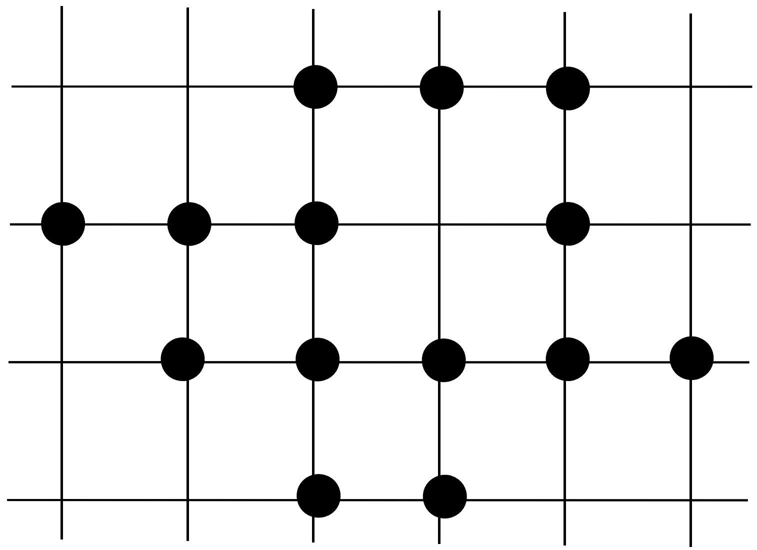
  ⇔ indicator fn $\delta_S : \mathbb{Z}^n \to \{0, +\infty\}$ is discretely convex

  ⇔ $\forall x, y \in S,\ \alpha \in [0,1], s \equiv \alpha x + (1-\alpha) y$

  $\exists z \in S$ s.t. $z(i) = \lfloor s(i) \rfloor \ or\ \lceil s(i) \rceil\ (\forall i)$



discretely convex



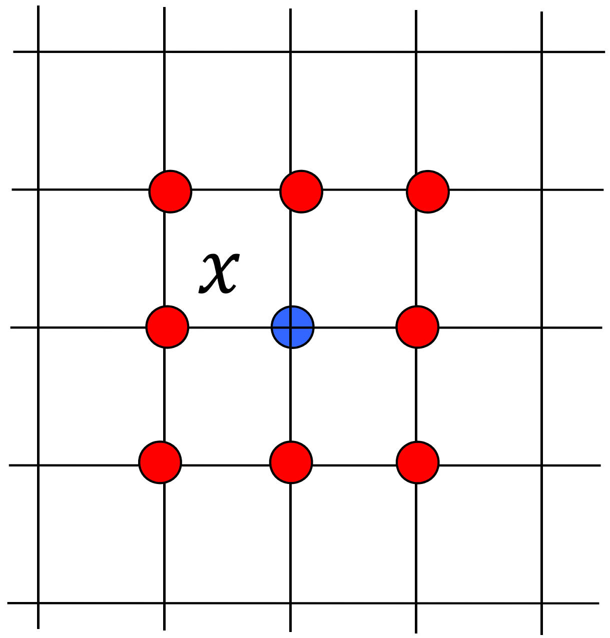
NOT discretely convex

Property of Discretely Convex Fn

- Thm: [local min = global min]

$x \in \arg\min\{f(z) \mid \ ||z - x||_\infty \leq 1\}$

$\Leftrightarrow x \in \arg\min\{f(z) \mid z \in \mathbb{Z}^n\}$



➔ validity of descent alg for minimization

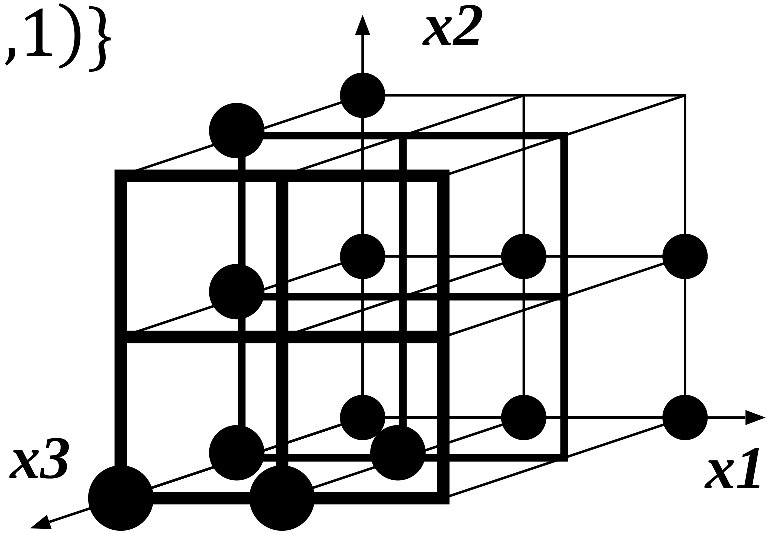
repeat: (i) find $z \in \mathrm{N}_\infty(x)$ with $f(z) < f(x)$

(ii) update $x := z$

※ size of neighborhood $\{z \mid ||z - x||_\infty \leq 1\}$ is 3^n --- exponential

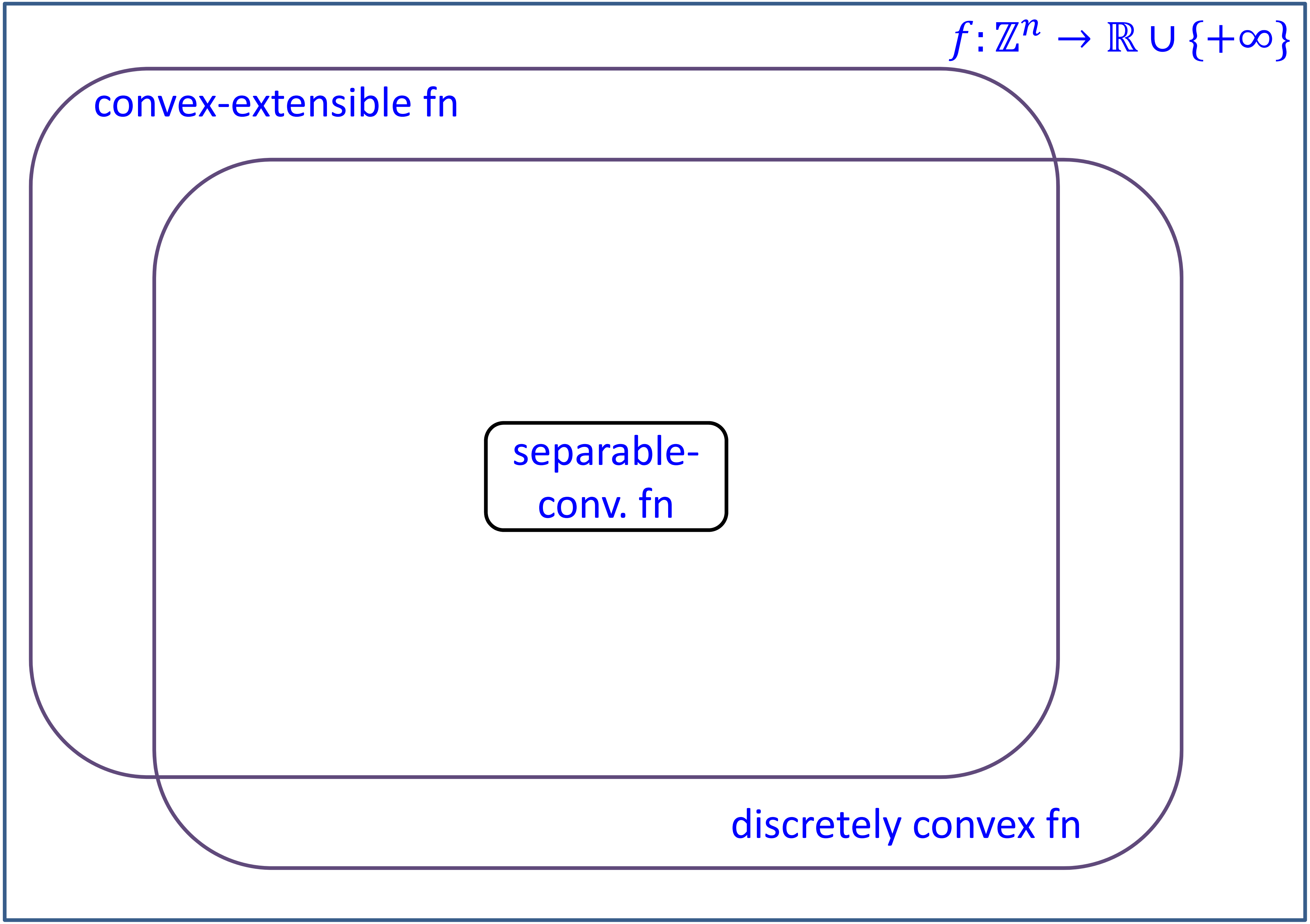
Bad Result of Discretely Convex Fn

- Fact: discretely conv fn is NOT convex-extensible
  discretely conv set is NOT convex-extensible
  (not satisfy "no-hole" condition)
- Example: $S = \{x \in \mathbb{Z}^3 | x_1 + x_2 + x_3 \leq 2, x_i \geq 0 (i = 1,2,3)\}$
  $\cup \{(1,2,0), (0,1,2), (2,0,1)\}$



➔ S is discretely convex, but has a "hole"

$\{(1,2,0) + (0,1,2) + (2,0,1)\} / 3 = (1,1,1) \notin S$


$f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$
convex-extensible fn
separable-conv. fn
discretely convex fn

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Integrally Convex Function

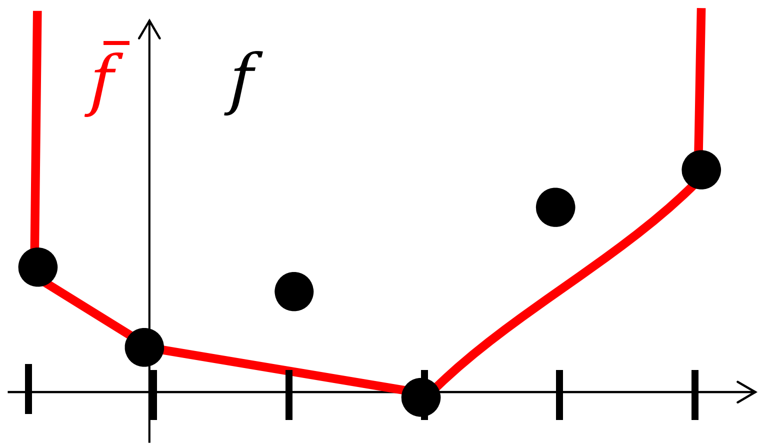
Convex Closure of Discrete Fn

- Def: convex closure $\bar{f}: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of $f: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$

  --- point-wise maximal convex fn satisfying $\bar{f}(y) \le f(y) \ (\forall y \in \mathbb{Z}^n)$

$$\bar{f}(x) = \min\{\textstyle\sum_{y \in \mathrm{dom}\, f} \alpha_y f(y) \mid \alpha_y \ge 0 \ (y \in \mathrm{dom}\, f), \sum_y \alpha_y = 1, \sum_y \alpha_y y = x\}$$

  - convex closure is convex fn



Local Convex Closure of Discrete Fn

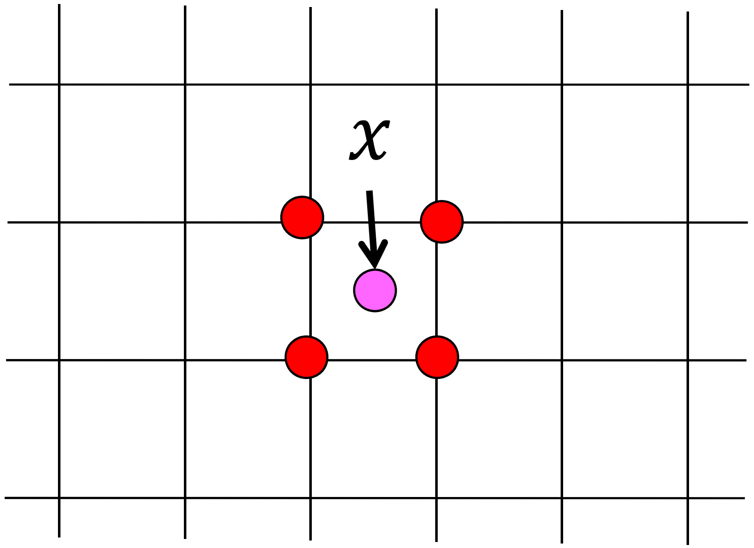
- Def: local convex closure $\tilde{f}: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of $f: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$

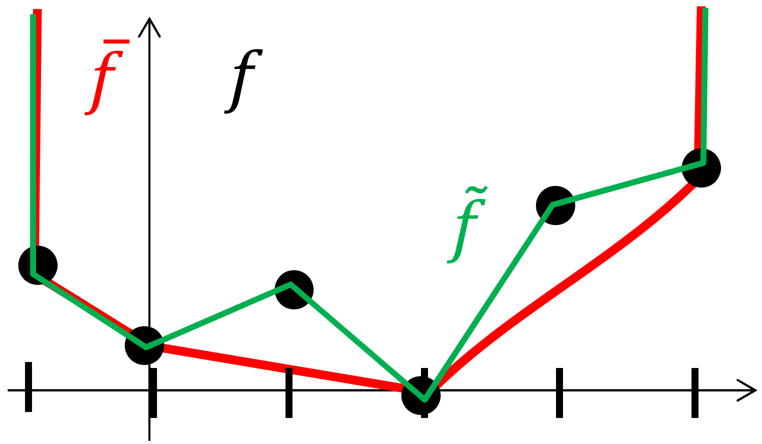
←→ collection of conv. closure on each hypercube $\tilde{f}(x) = \min\{\sum_{y \in \mathrm{HC}(x)} \alpha_y f(y)$

$$| \alpha_y \geq 0 \; (y \in \mathrm{HC}(x)), \sum_y \alpha_y = 1, \sum_y \alpha_y y = x\}$$

$$\mathrm{HC}(x) = \{y \in \mathbb{Z}^n \mid y(i) = \lfloor x(i) \rfloor or \lceil x(i) \rceil \; (\forall i)\}$$

  - $\tilde{f}(x) = f(x) \; (\forall x \in \mathbb{Z}^n)$
  - local convex closure $\tilde{f}$ is not convex

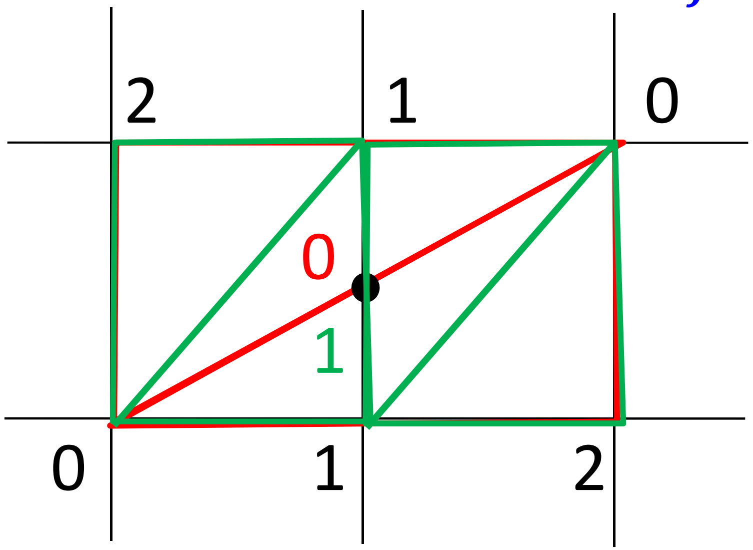




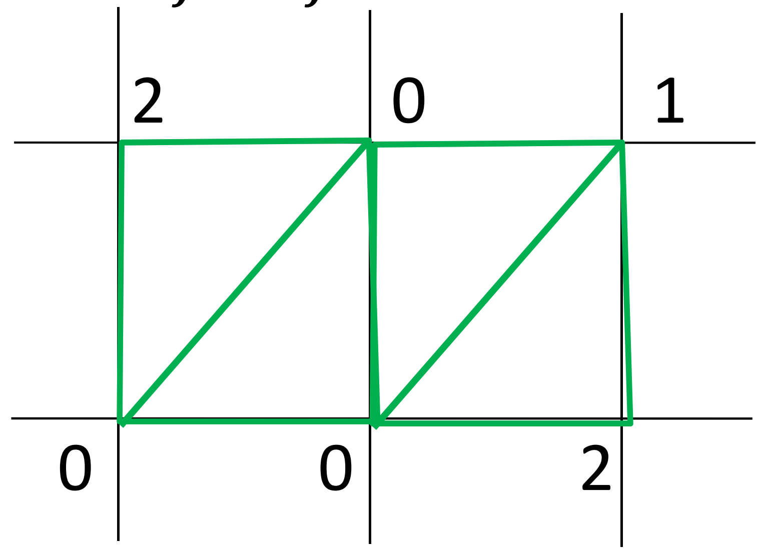
Definition of Integrally Convex Fn

- Def: $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is integrally convex (Favati-Tardella 1990)

⇔ local conv. closure $\tilde{f}$ is convex fn ⇔ $\tilde{f} = \bar{f}$



convex-extensible
but NOT integrally convex



convex-extensible
& integrally convex

Properties of Integrally Convex Fn

- by definition, integrally convex fn is
  - convex-extensible
  - discretely convex

    ➔ local min w.r.t. $\{z \mid ||z - x||_\infty \leq 1\}$ = global min

Bad Results of Integrally Convex Fn

- by definition, integrally convex fn is
  - convex-extensible
  - discretely convex

    ➔ local min w.r.t. $\{z \mid ||z - x||_\infty \leq 1\}$ = global min

    but, neighborhood contains 3^n vectors (exponential)

- any function f with $\mathrm{dom}\, f = \{0,1\}^n$ is integrally convex

  ➔ no good structure

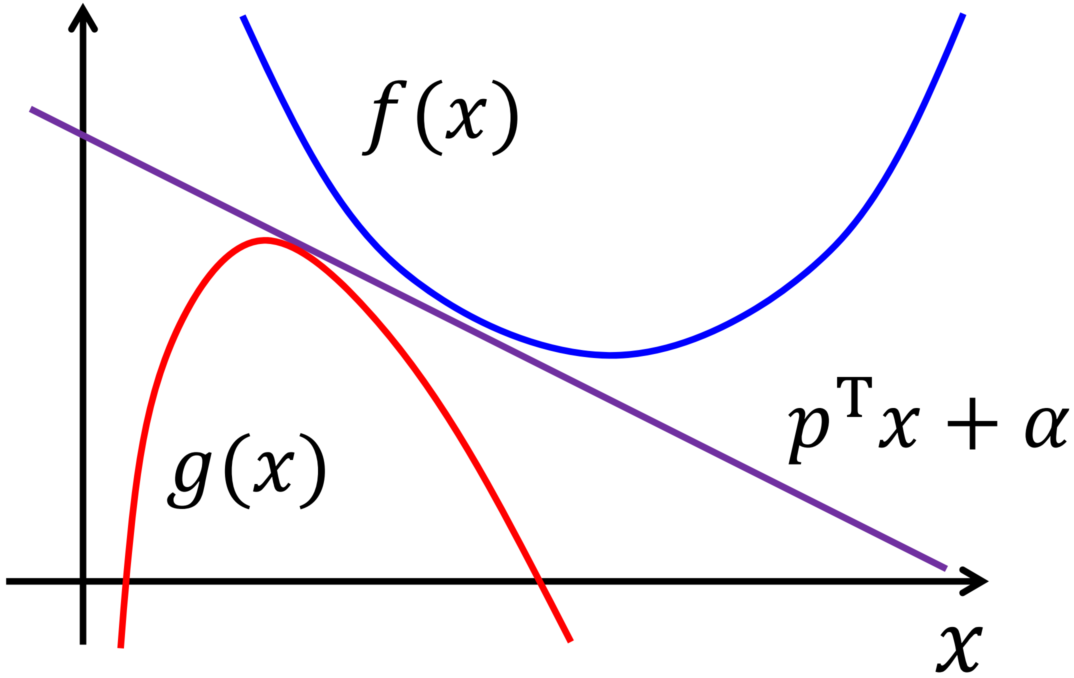
- failure of "discrete" separation theorem

Separation Theorem for Convex Fn

- Separation Theorem:

  f: convex fn, g: concave fn, $f(x) \geq g(x)$ $(\forall x \in \mathbb{R}^n)$

  ➔ ∃ affine fn $p^T x + \alpha$ s.t. $f(x) \geq p^T x + \alpha \geq g(x)$ $(\forall x \in \mathbb{R}^n)$



  - equivalent to Duality Theorem for nonlinear programming

    ➔ efficient primal-dual-type algorithm

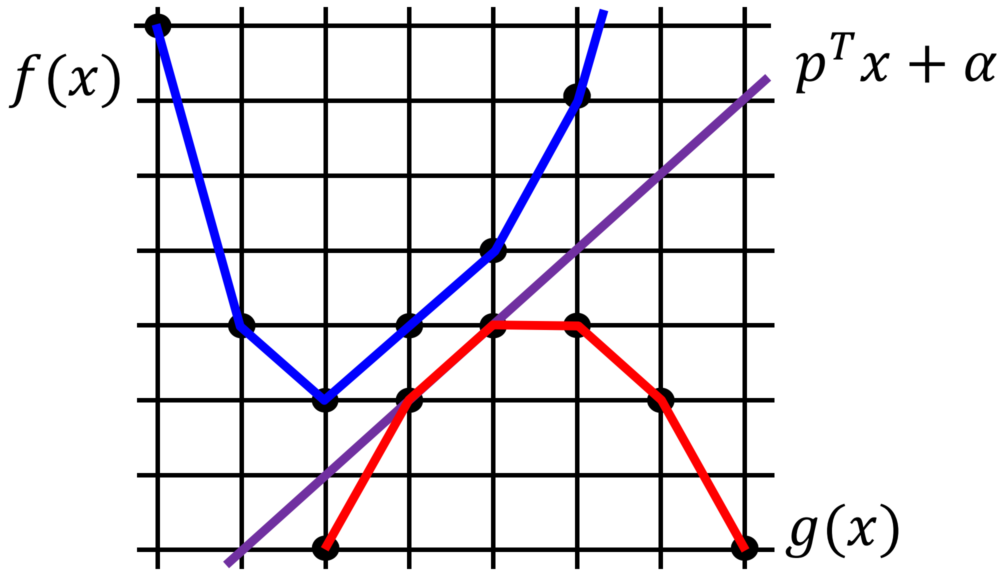
Discrete Separation Thm for Discrete Convex Fn

- "Discrete" Separation Theorem:

  f: "discrete convex" fn, g: "discrete concave" fn,

  $$f(x) \geq g(x) \ (\forall x \in \mathbb{Z}^n)$$

  ➔ $\exists$ affine fn $ax + b$ s.t. $f(x) \geq ax + b \geq g(x)$ $(\forall x \in \mathbb{Z}^n)$



- equivalent to Duality Theorem for combinatorial optimization

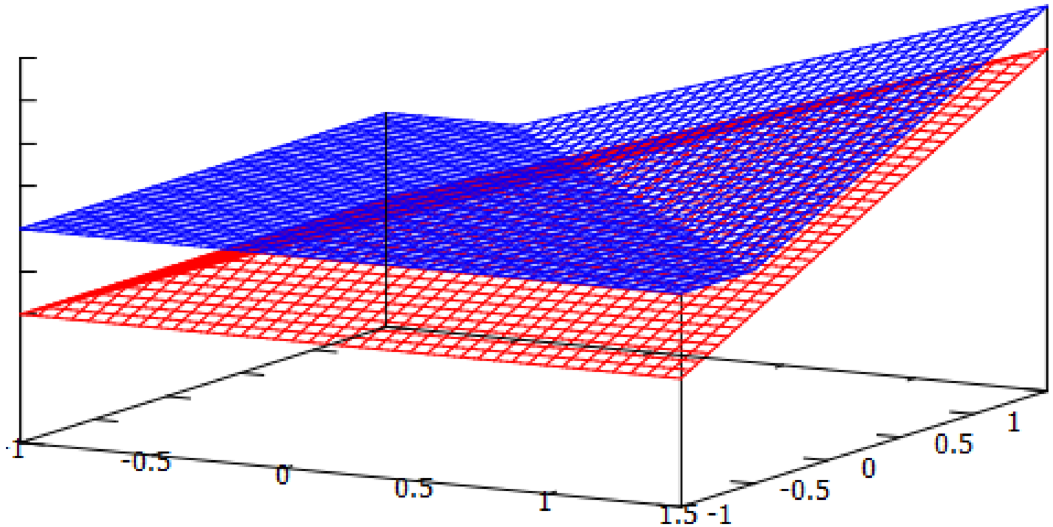
  ➔ efficient primal-dual-type algorithm

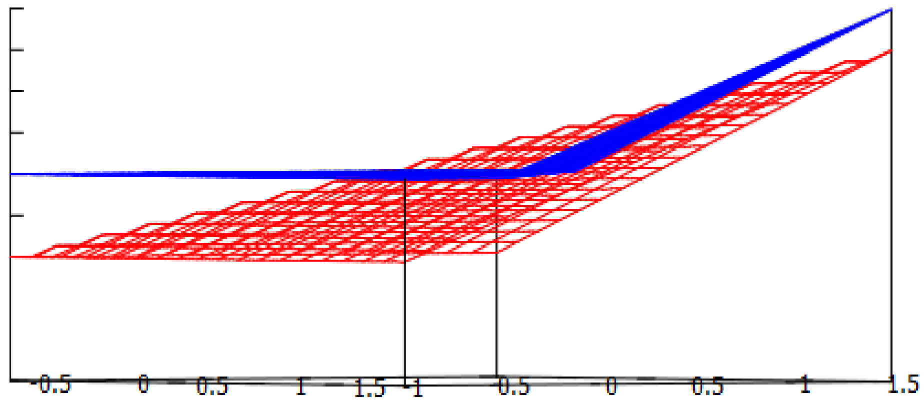
Failure of Discrete Separation for Integrally Convex/Concave Fns

- $\exists f$: integrally convex, g: integrally concave s.t.

$$f(x) \geq g(x) \ (\forall x \in \mathbb{Z}^n)$$

but $\nexists$ affine fn $p^T x + \alpha$ with $f(x) \geq p^T x + \alpha \geq g(x) \ (\forall x \in \mathbb{Z}^n)$



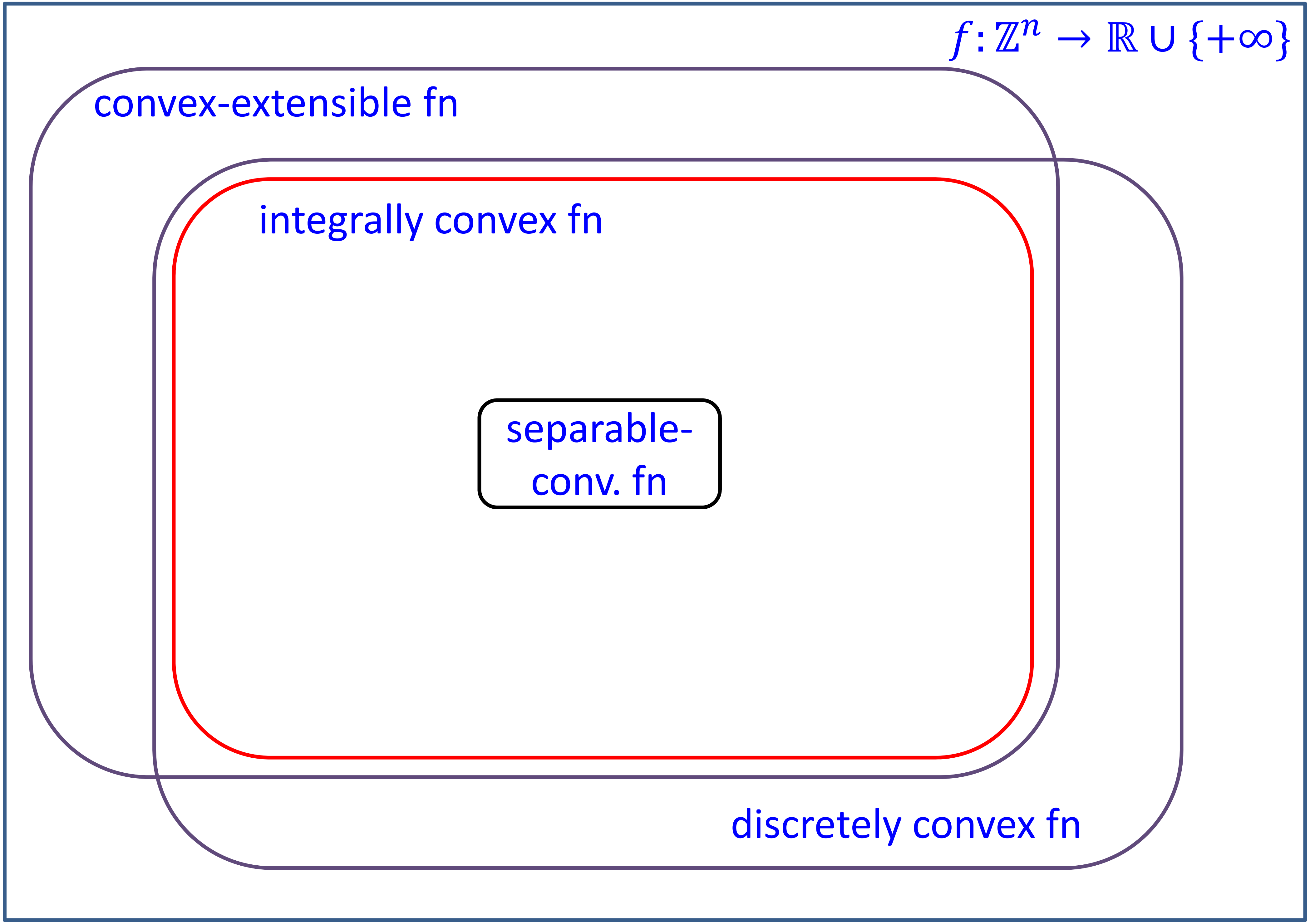


$f(x_1, x_2) = \max\{0, x_1 + x_2 - 1\}$ --- integrally convex,

$g(x_1, x_2) = \min\{x_1, x_2\}$ --- integrally concave,

$f(x_1, x_2) \geq g(x_1, x_2) \ (\forall (x_1, x_2) \in \mathbb{Z}^2)$, but $f(0.5, 0.5) < g(0.5, 0.5)$

➔ no affine fn with $f(x) \geq p^T x + \alpha \geq g(x) \ (\forall x \in \mathbb{Z}^2)$


$f: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$
convex-extensible fn
integrally convex fn
separable-conv. fn
discretely convex fn

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L-convex Function

Definition of L♮-convex Fn

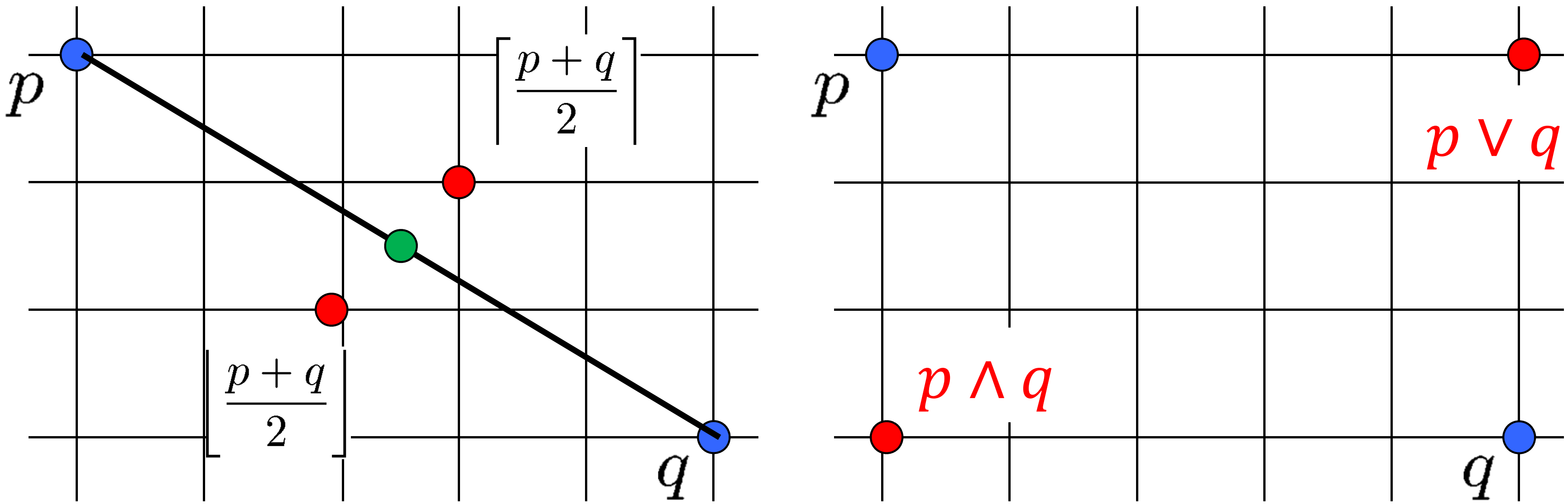
- L♮ -- L-natural, L=Lattice
- Def: $g: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is L♮-convex (Fujishige-Murota 2000)

⇔ [discrete mid-point convexity]

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right) \quad (\forall p, q \in \mathbb{Z}^n)$$

⇔ integrally convex + submodular (Favati-Tardella 1990)

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q) \quad (\forall p, q \in \mathbb{Z}^n)$$



※ L♮-convex → int. convex → conv.-extensible & discr. convex

Examples of L♮-convex Fn

- univariate convex $\varphi: \mathbb{Z} \to \mathbb{R}$ ←→ $\varphi(t-1) + \varphi(t+1) \geq 2\varphi(t)$
- separable-convex fn
- submodular set fn ←→ L♮-conv fn with $\mathrm{dom}\ g = \{0,1\}^n$
- quadratic fn $g(p) = p^T A p$ is L♮-convex
  ←→ $a_{ij} \leq 0\ (i \neq j)$, $\sum_j a_{ij} \geq 0$

$$\begin{bmatrix} 4 & & & -1 \\ & 3 & -2 & \\ & -2 & 3 & -1 \\ -1 & & -1 & 5 \end{bmatrix}$$

- Range: $g(p) = \max\{p_1, p_2, \ldots, p_n\} - \min\{p_1, p_2, \ldots, p_n\}$
- min-cost tension problem

$$g(p) = \sum_{i=1}^{n} \varphi_i(p_i) + \sum_{i,j} \psi_{ij}(p_i - p_j)$$

(φ_i, ψ_{ij}: univariate discrete conv fn)

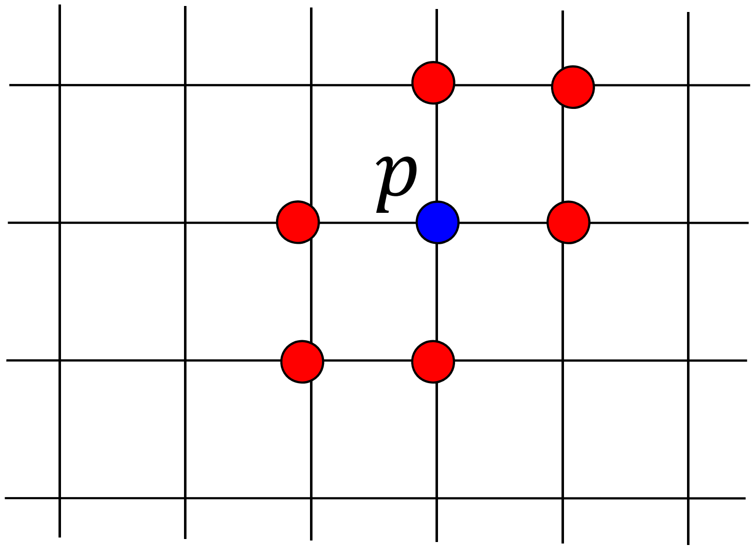
Optimality Condition by Local Property

- Thm: [local min = global min]

$$g(p) \leq \min\{g(p+\chi_X), g(p-\chi_X)\} \ (\forall X \subseteq \{1,2,\ldots,n\})$$

$$\Leftrightarrow g(p) \leq g(q) \ (\forall q \in \mathbb{Z}^n)$$

$$\chi_X(i) = \begin{cases} 1 & (i \in X) \\ 0 & (i \notin X) \end{cases}$$



※ local minimality check can be done efficiently

$$\rho(X) \equiv g(p+\chi_X), \mu(X) \equiv g(p-\chi_X)$$

➔ ρ, μ : submodular set fns, minimization in poly-time

M-convex Function

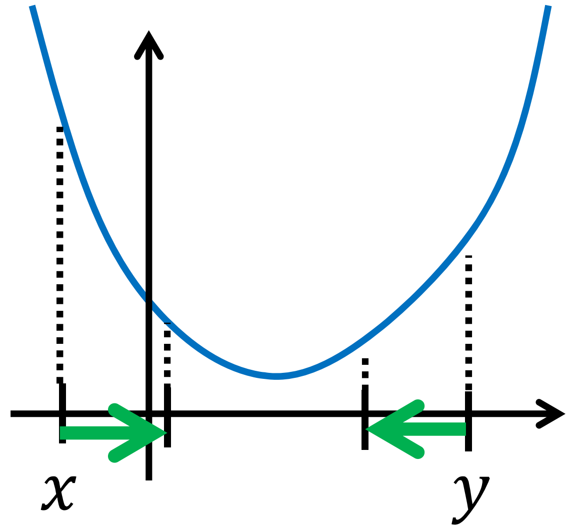
Characterization of Convex Function

- Prop: ["equi-distant" convexity]

$f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex ⇔ $\forall x, y \in \mathbb{R}^n,\ \exists \delta > 0,$

$$f(x) + f(y) \geq f\big(x - \alpha(x - y)\big) + f\big(y + (\alpha(x - y)\big)$$

$$(0 \leq \forall \alpha \leq \delta)$$



Definition of M♮-convex Function

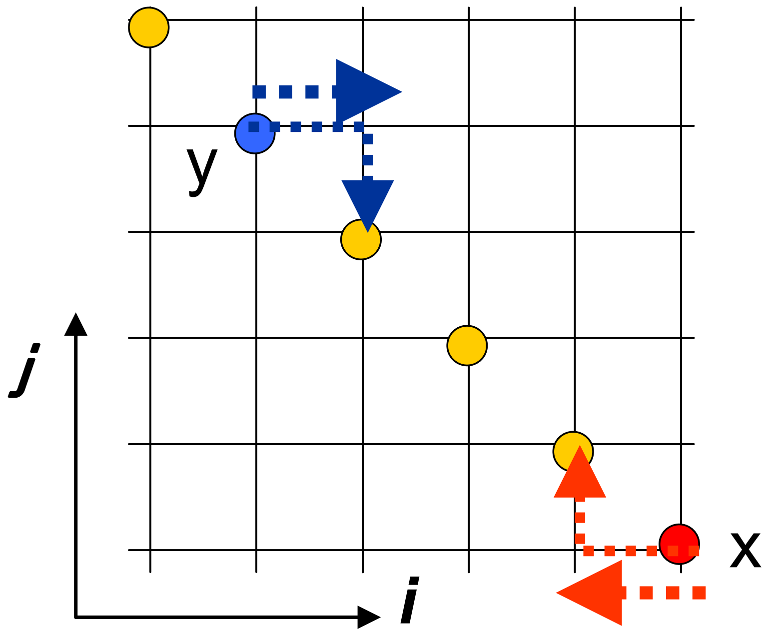
M=Matroid

Def: $f: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is M♮-convex (Murota-Shioura99)

⟺ $\forall x, y \in \mathbb{Z}^n, \forall i: x(i) > y(i)$:

(i) $f(x) + f(y) \geq f(x - \chi_i) + f(y + \chi_i)$, or

(ii) $\exists j: x(j) < y(j)$ s.t. $f(x) + f(y) \geq f\left(x - \chi_i + \chi_j\right) + f(y + \chi_i - \chi_j)$



Examples of M$^\natural$-convex Functions

- Univariate convex $\varphi: \mathbb{Z} \to \mathbb{R}$ ⟺ $\varphi(t-1) + \varphi(t+1) \geq 2\varphi(t)$
- Separable convex fn on polymatroid:

For integral polymatroid $P \subseteq \mathbb{Z}_+^n$ and univariate convex φ_i

$$f(x) = \sum_{i=1}^{n} \varphi_i(x(i)) \quad \text{if} \quad x \in P$$

- Matroid rank function [Fujishige05]

$f(X) = \max\{|Y| \mid Y: \text{independent set}, Y \subseteq X\}$ is M$^\natural$-concave

- Weighted rank function [Shioura09] $(w \geq 0)$

$f(X) = \max\{w(Y) \mid Y: \text{independent set}, Y \subseteq X\}$ is M$^\natural$-concave

- Gross substitutes utility in math economics/game theory

⟺ M$^\natural$-concave fn on $\{0,1\}^n$ [Fujishige-Yang03]

Properties of M♮-convex Fn

- Thm: [local min = global min]

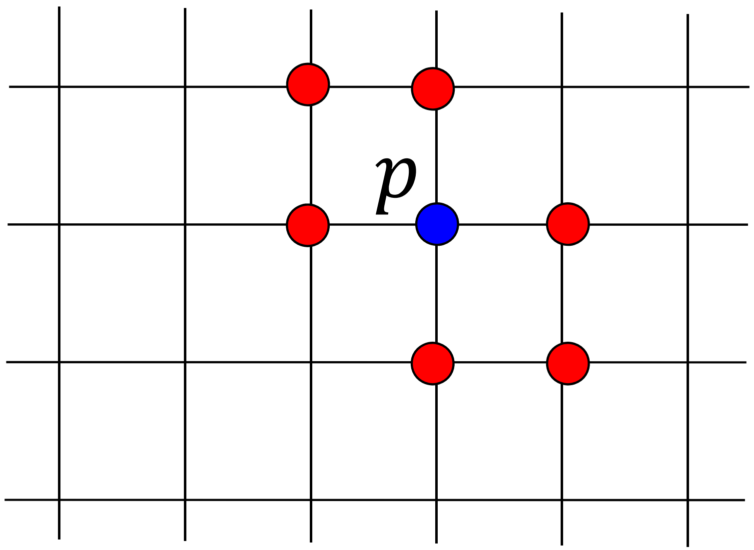
$f(x) \le f(x \pm \chi_j)\ (\forall j),$

$f(x) \le f(x + \chi_j - \chi_k)\ (\forall j, k),$

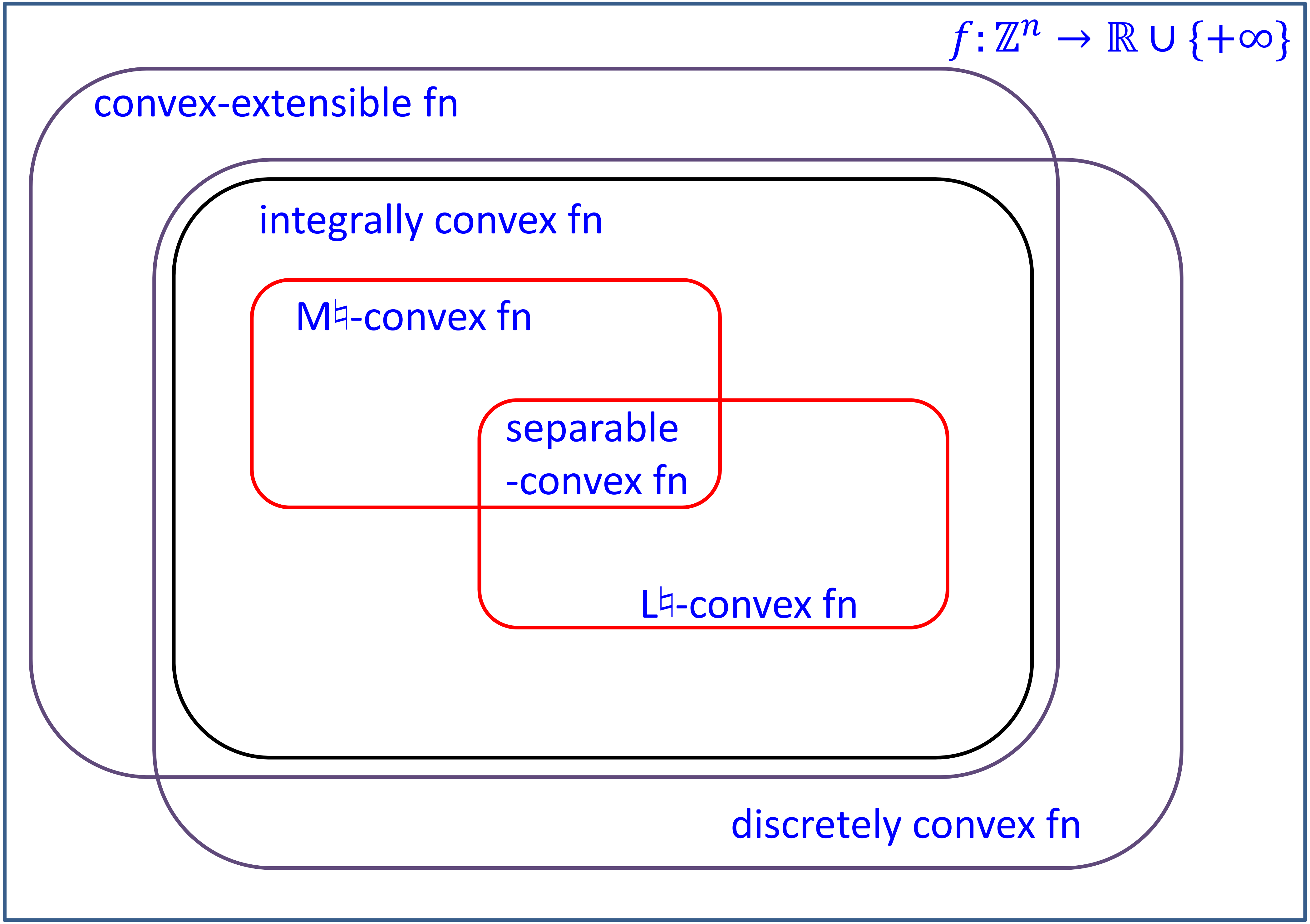
$\Leftrightarrow f(x) \le f(y)\ (\forall y \in \mathbb{Z}^n)$

$$\chi_j(i) = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

※ size of neighborhood = $O(n^2)$



- M♮-convex ➔ int. convex ➔ conv.-extensible & discr. convex


$f: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$
convex-extensible fn
integrally convex fn
M♮-convex fn
separable
-convex fn
L♮-convex fn
discretely convex fn

Outline of Talk

- Overview of Discrete Convex Analysis
- Desirable Properties of Discrete Convexity
- convex-extensible fn
- Miller’s discretely convex fn
- Favati-Tardella’s integrally convex fn
- M-convex & L-convex fns
- duality and conjugacy theorems for discrete convex fn

Conjugacy and Duality

Conjugacy for Convex Functions

- Legendre transformation for $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$:

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbb{R}^n\} \quad (p \in \mathbb{R}^n)$$

convex fn is closed under Legendre transformation

- Thm:

f: convex ➔ $f^\bullet$:convex, $\left(f^\bullet\right)^\bullet = f$ (if f is closed)

Conjugacy for L-/M-Convex Functions

- integer-valued fn $f: \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$
- discrete Legendre transformation:

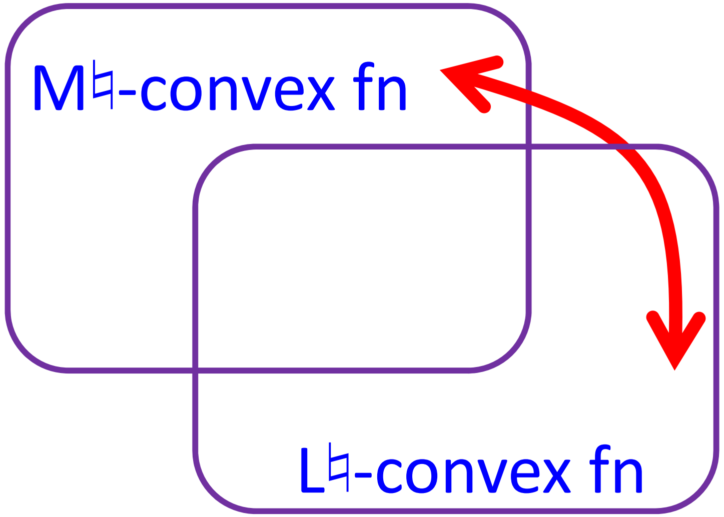
$$f^{\bullet}(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbb{Z}^n\} \quad (p \in \mathbb{Z}^n)$$

L♮-convex fn and M♮-convex fn are conjugate

- Thm:

(i) f: M♮-conv ➔ $f^{\bullet}$: L♮-conv, $(f^{\bullet})^{\bullet} = f$

(ii) f: L♮-conv ➔ $f^{\bullet}$: M♮-conv, $(f^{\bullet})^{\bullet} = f$



- generalization of relations in comb. opt.:
  - matroid ←→ rank fn [Whitney 35]
  - polymatroid ←→ submodular fn [Edmonds 70]

Fenchel Duality Theorem for Convex Fn

- Legendre transformation:

$$f^{\bullet}(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbb{R}^n\}$$
$$f^{\circ}(p) = \inf\{\langle p, x\rangle - f(x) \mid x \in \mathbb{R}^n\}$$

- Fenchel Duality Thm:

f: convex, g: concave ➔

$$\inf_{x\in\mathbb{R}^n}\{f(x) - g(x)\} = \sup_{p\in\mathbb{R}^n}\{g^{\circ}(p) - f^{\bullet}(p)\}$$

($f^{\bullet}$: convex, g°: concave)

- Fenchel duality thm ⇔ separation theorem

Fenchel-Type Duality Theorem for L-/M-Convex Fn

- discrete Legendre transformation:

$$f^{\bullet}(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbb{Z}^n\}$$
$$f^{\circ}(p) = \inf\{\langle p, x\rangle - f(x) \mid x \in \mathbb{Z}^n\}$$

- Fenchel-type Duality Thm: [Murota 96,98]

f: M♮-convex, g: M♮-concave ➔

$$\inf_{x \in \mathbb{Z}^n} \{f(x) - g(x)\} = \sup_{p \in \mathbb{Z}^n} \{g^{\circ}(p) - f^{\bullet}(p)\}$$

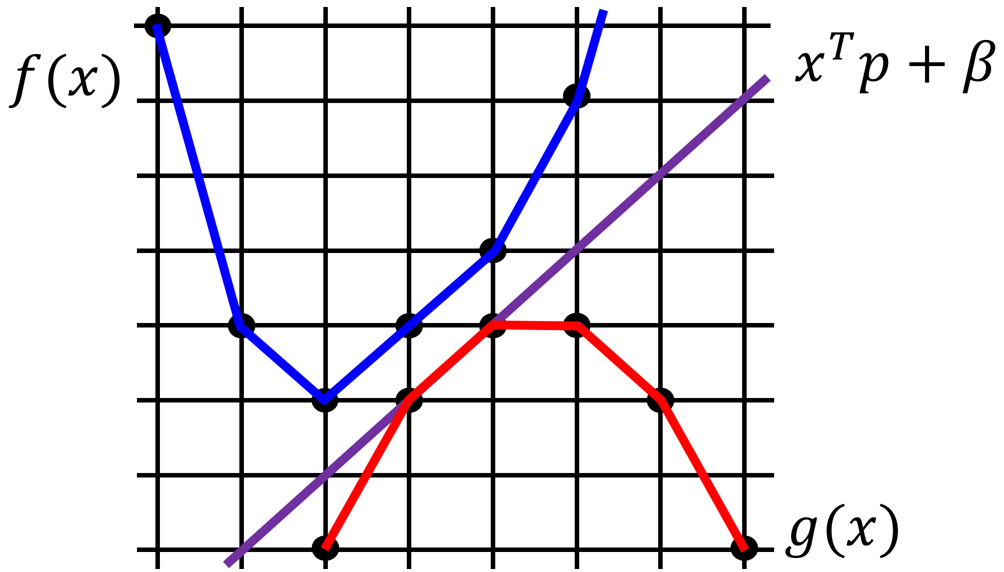
($f^{\bullet}$: L♮-convex, g°: L♮-concave)

Discrete Separation Thm for L♮ Convex Fn

- L♮ Separation Theorem: [Murota 96,98]

  f: L♮-convex fn, g: L♮-concave fn, $f(p) \geq g(p)$ $(\forall p \in \mathbb{Z}^n)$

  ➔ ∃ affine fn s.t. $f(p) \geq x^T p + \beta \geq g(p)$ $(\forall p \in \mathbb{Z}^n)$

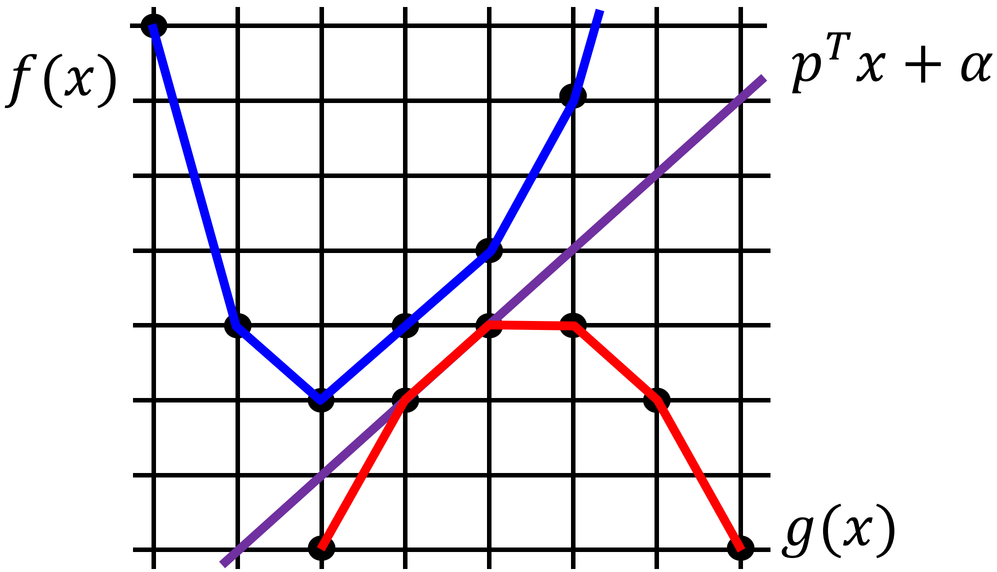


Discrete Separation Thm for M♮ Convex Fn

- M♮ Separation Theorem: [Murota 96,98]

  f: M♮-convex fn, g: M♮-concave fn, $f(x) \geq g(x)$ $(\forall x \in \mathbb{Z}^n)$

  ➔ ∃ affine fn s.t. $f(x) \geq p^T x + \alpha \geq g(x)$ $(\forall x \in \mathbb{Z}^n)$



Relation among Duality Thms

M♮ Separation Thm

$$f(x) \geq p^T x + \alpha \geq g(x)$$

⇕

Fenchel-type Duality Thm

$$\inf\{f(p) - g(p)\} = \sup\{g^\circ(x) - f^\bullet(x)\}$$

⇕

L♮ Separation Thm

$$f^\bullet(p) \geq x^T p + \beta \geq g^\circ(p)$$

⇒ weight splitting thm for weighted matroid intersection [Iri-Tomizawa 76, Frank 81]

⇒ (poly)matroid intersection thm [Edmonds 70]
weighted matroid intersection thm [Iri-Tomizawa 76, Frank 81]
Fenchel-type duality thm for subm. fn [Fujishige 84]

⇒ discrete separation for subm. fn [Frank 82]